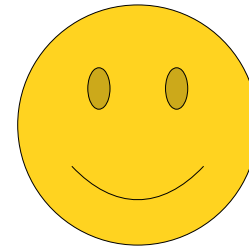
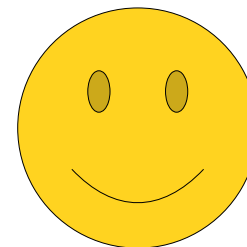


Guide to Cantor's Theorem

Hi everybody!



In this guide, I'd like to talk about a formal proof of Cantor's theorem, the diagonalization argument we saw in our very first lecture.



If S is a set, then $|S| < |\mathcal{P}(S)|$

Here's the statement of Cantor's theorem that we saw in our first lecture. It says that every set is strictly smaller than its power set.



If S is a set, then $|S| < |\wp(S)|$

Our goal will be to rigorously prove this statement.



If S is a set, then $|S| < |\wp(S)|$

Before we can do that, though, we need to address one big unresolved question.



If S is a set, then $|S| < |\wp(S)|$

In Friday's lecture, we talked about what it means for two sets to have equal cardinality and for two sets to not have equal cardinality.



If S is a set, then $|S| < |\wp(S)|$

specifically, we said that $|S| = |T|$ if
there is a bijection $f : S \rightarrow T$.



If S is a set, then $|S| < |\wp(S)|$

As a result, $|S| \neq |T|$ if there are no bijections at from S to T .



If S is a set, then $|S| < |\wp(S)|$

If we want to prove that two sets do have the same cardinality, we can do so by finding a bijection between them.



If S is a set, then $|S| < |\wp(S)|$

If we want to prove that two sets have different cardinalities, we need to show that there's no possible bijection between them.



If S is a set, then $|S| < |\wp(S)|$

However, take a look at the statement
of Cantor's theorem.



If S is a set, then $|S| < |\wp(S)|$

In this statement, we're saying that $|S|$ is strictly smaller than $|\wp(S)|$, but we don't yet have a definition for what that means!



If S is a set, then $|S| < |\wp(S)|$

In other words, we can't even start proving this statement yet, since we never said what it means for one set to be strictly smaller than one another!

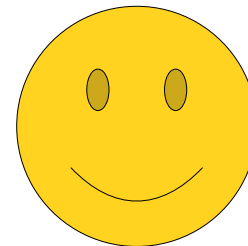


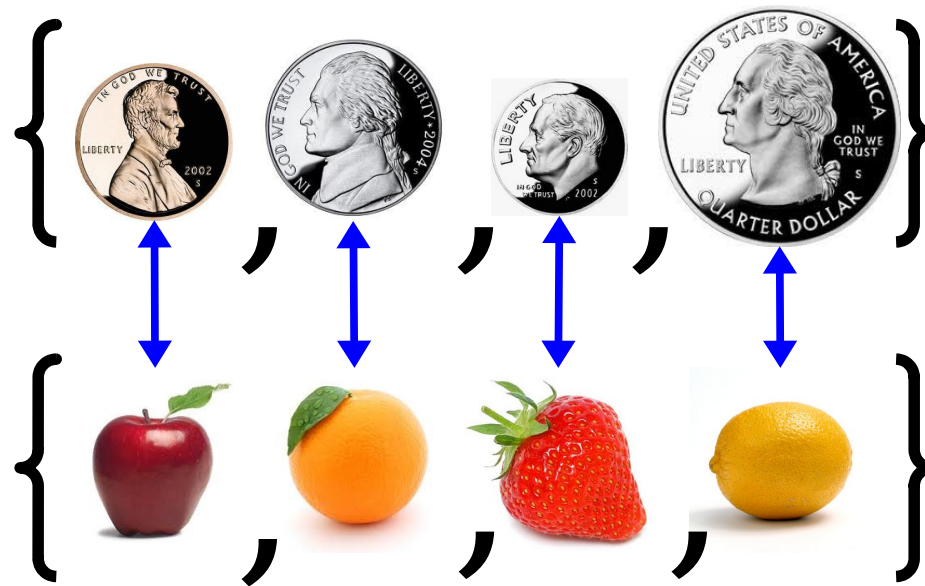
If S is a set, then $|S| < |\wp(S)|$

So, before we take a stab at proving Cantor's theorem, let's first go and make sure we have a definition for how to rank set cardinalities.



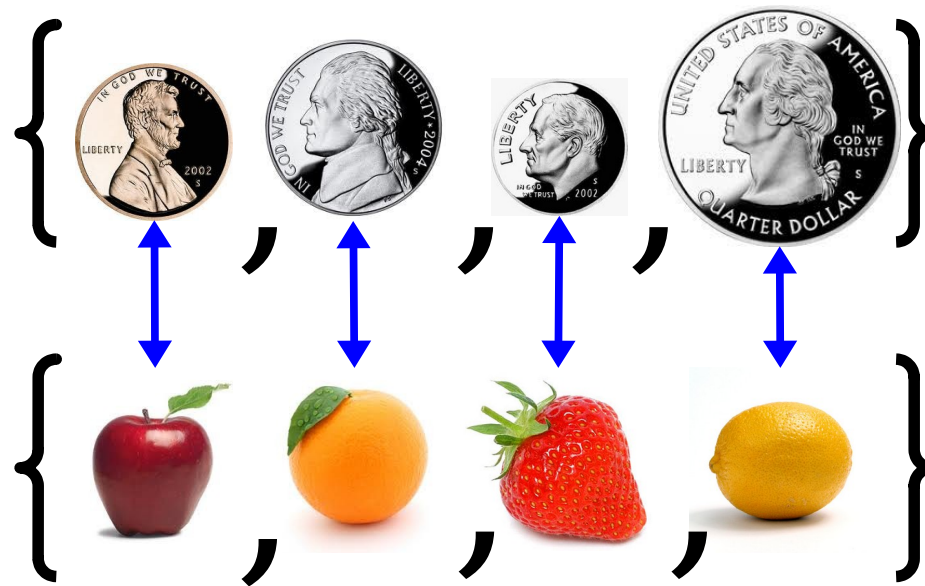
If you'll remember, when first talked about equal cardinalities way back in our first lecture, we said that two sets have the same cardinality if we can pair off the elements of the sets with no elements uncovered.





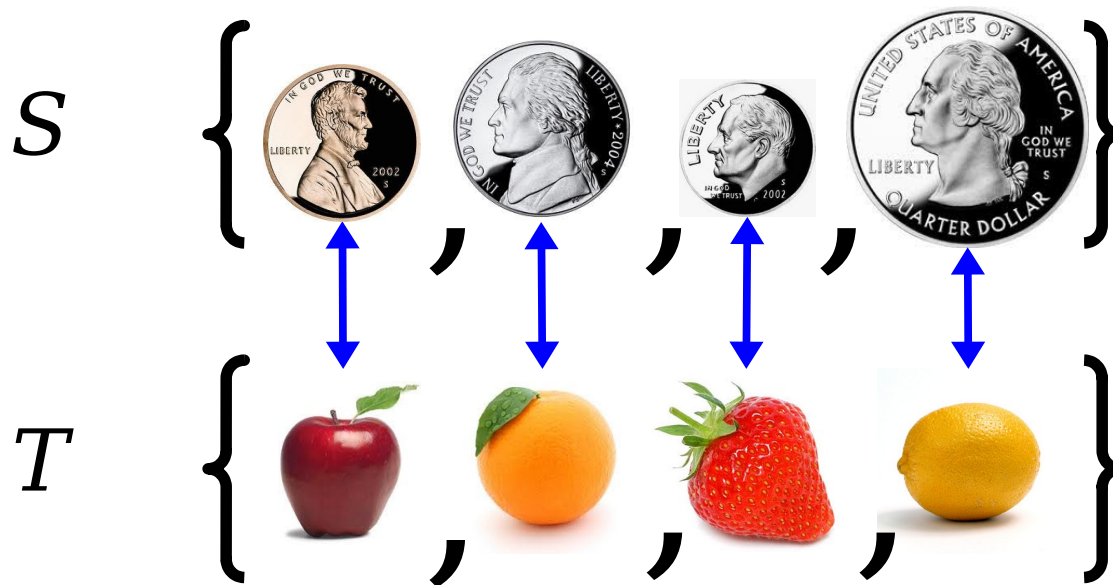
That's why these two sets have the same cardinality.





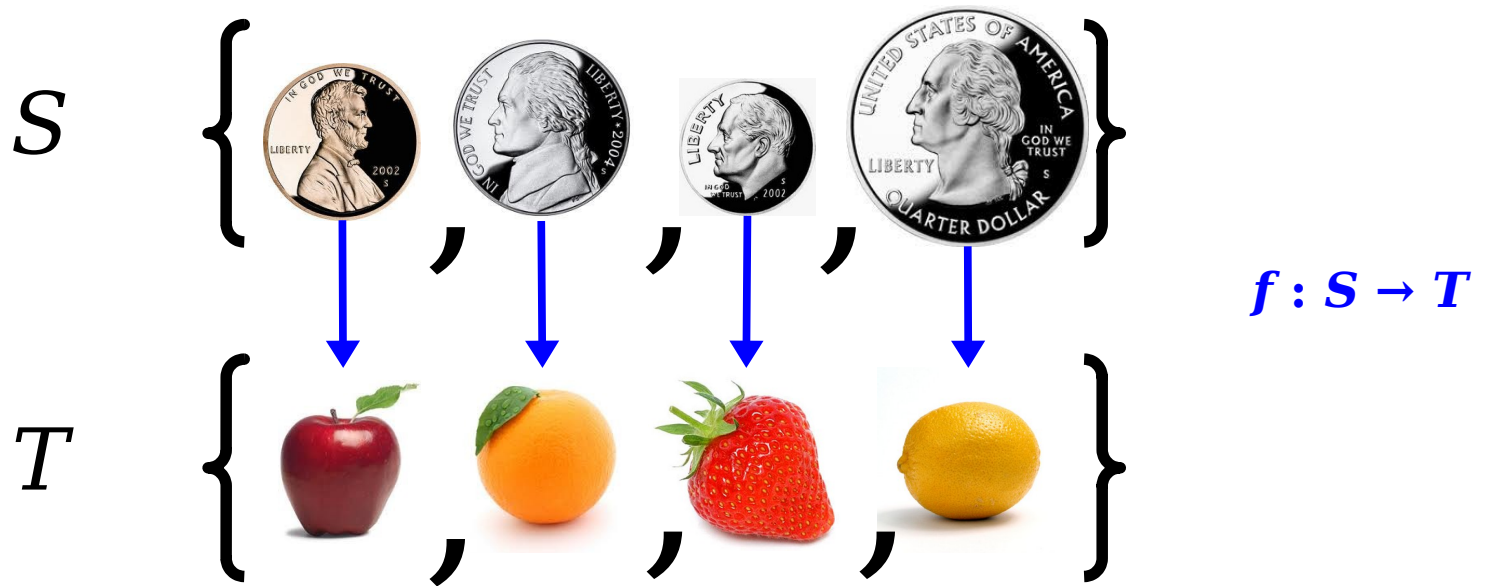
Once we had the language of bijections,
we were able to give a more formal
definition of what "pairing things off"
means.





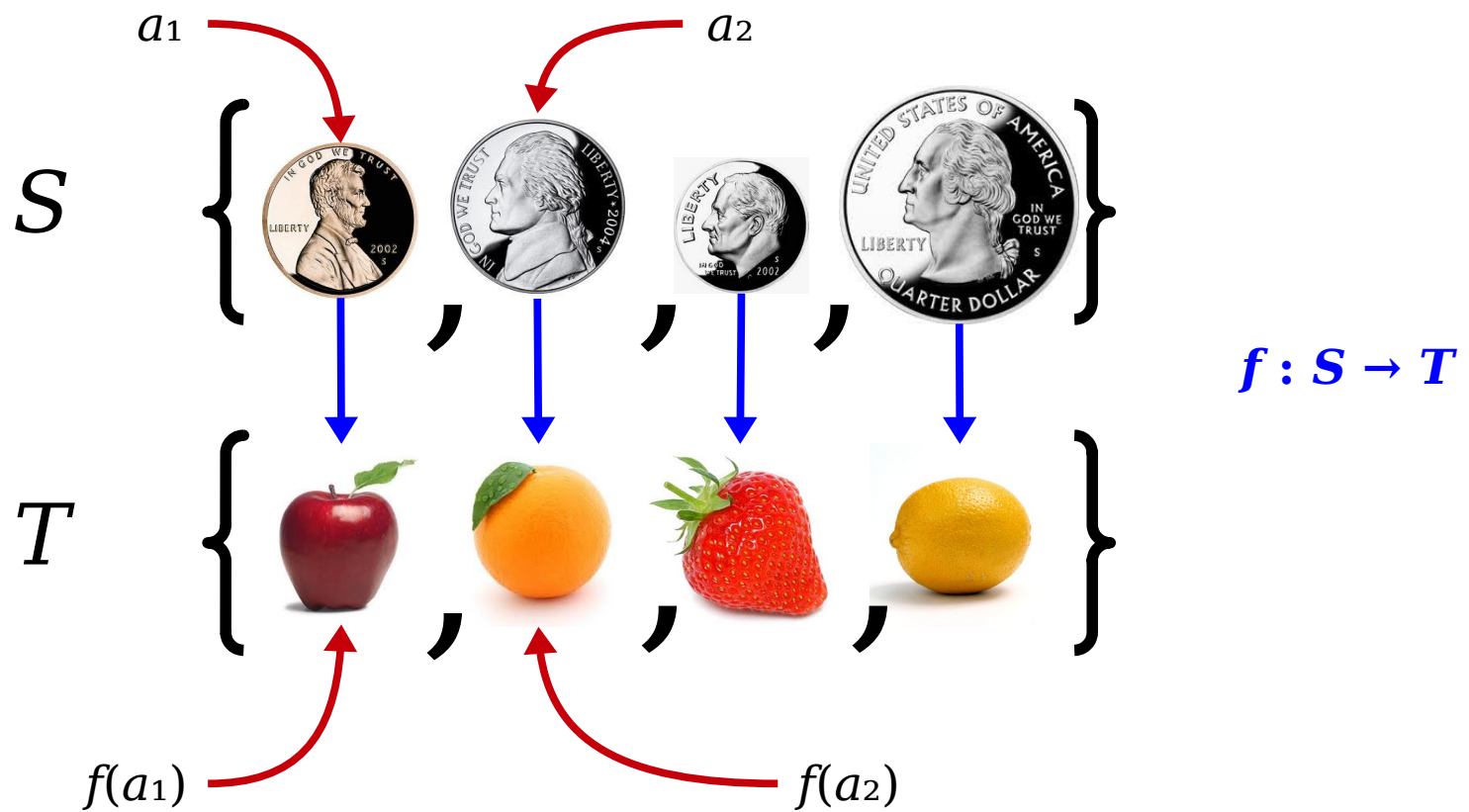
specifically, if we label these two
sets S and T ...





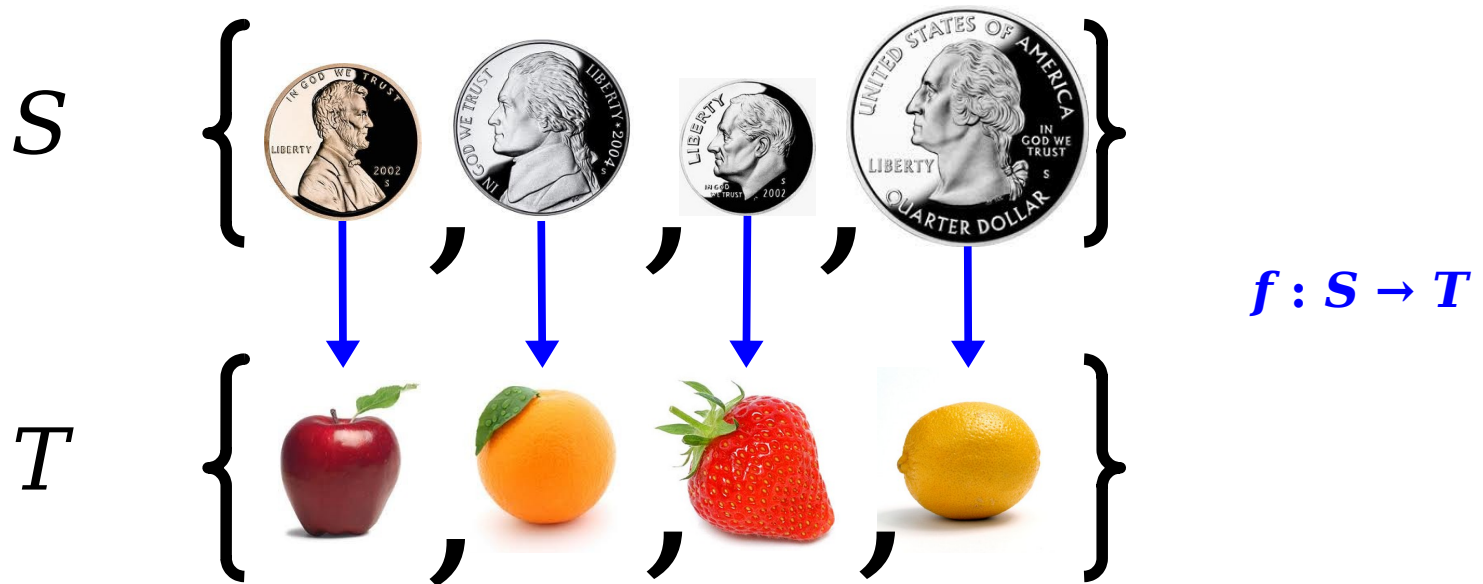
We can think of the pairing as a function from S to T .





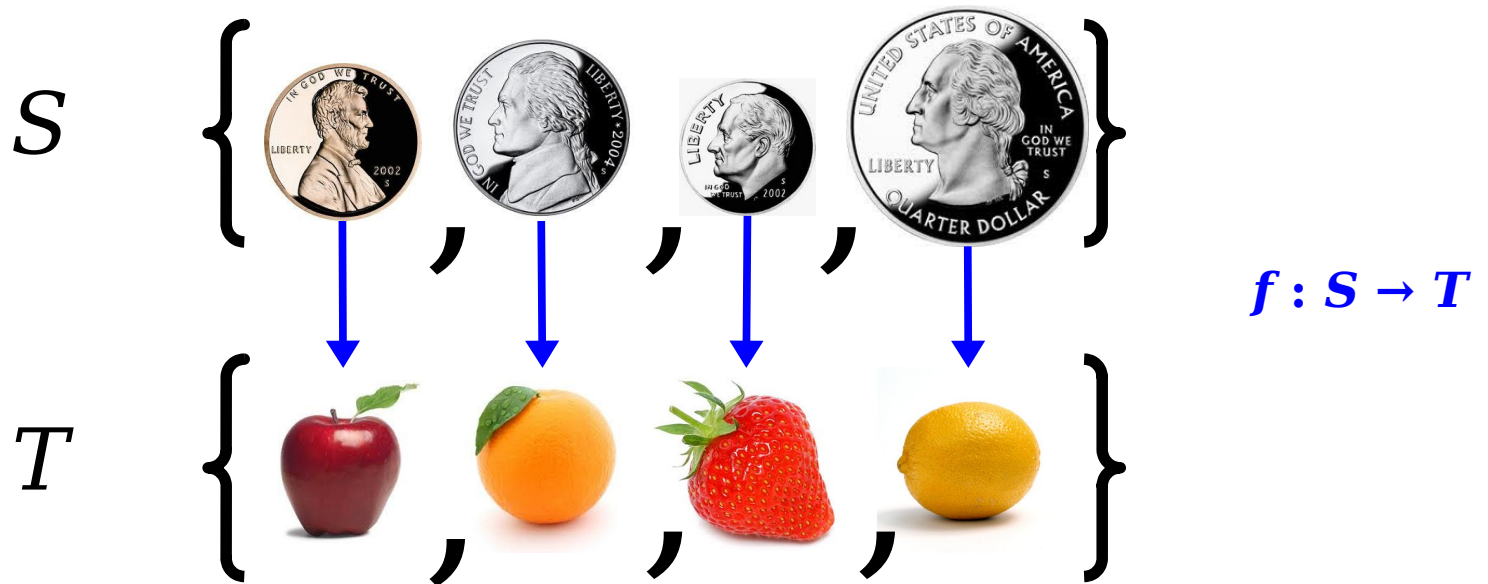
This function will be injective - any two different inputs produce different outputs.





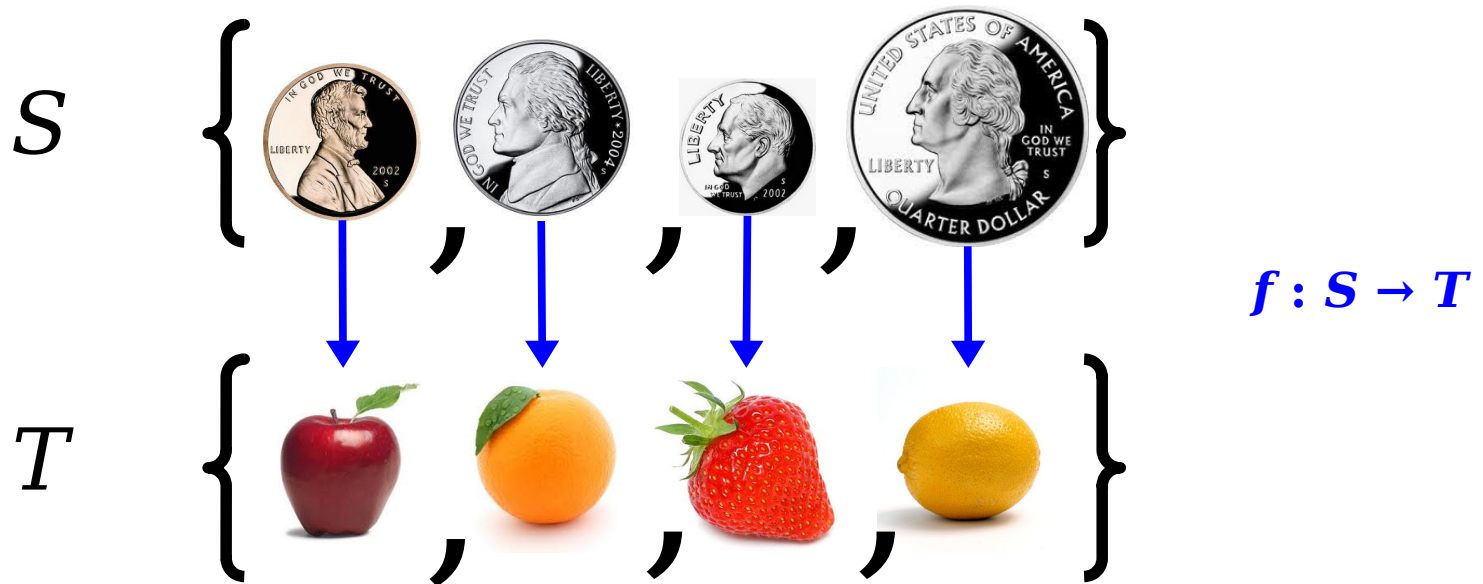
It's also surjective, since any element of the codomain (here, T) will be paired with something in S .





That's where we get the bijection-based definition of equal set cardinality from.



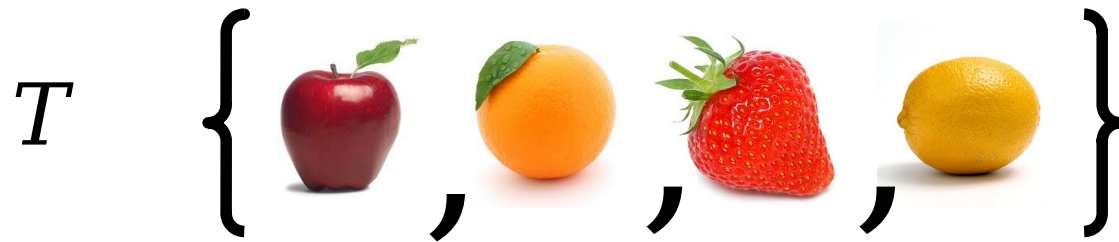


Could we develop some sort of analogous definition for what it means for one set to be "at least as big" as another?



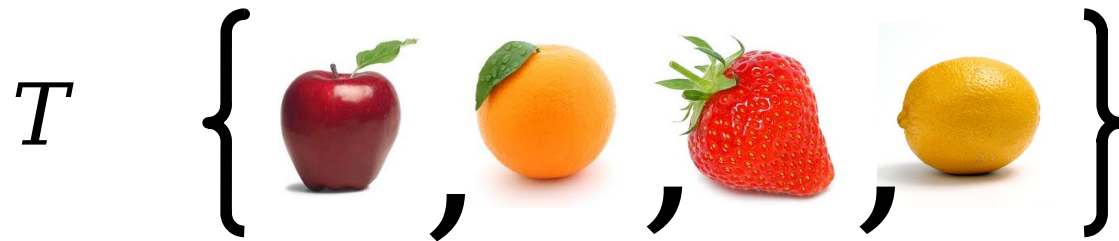
Let's begin by picking two new sets...



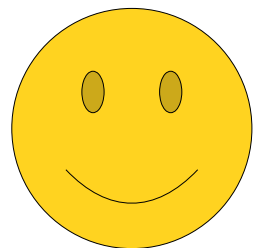


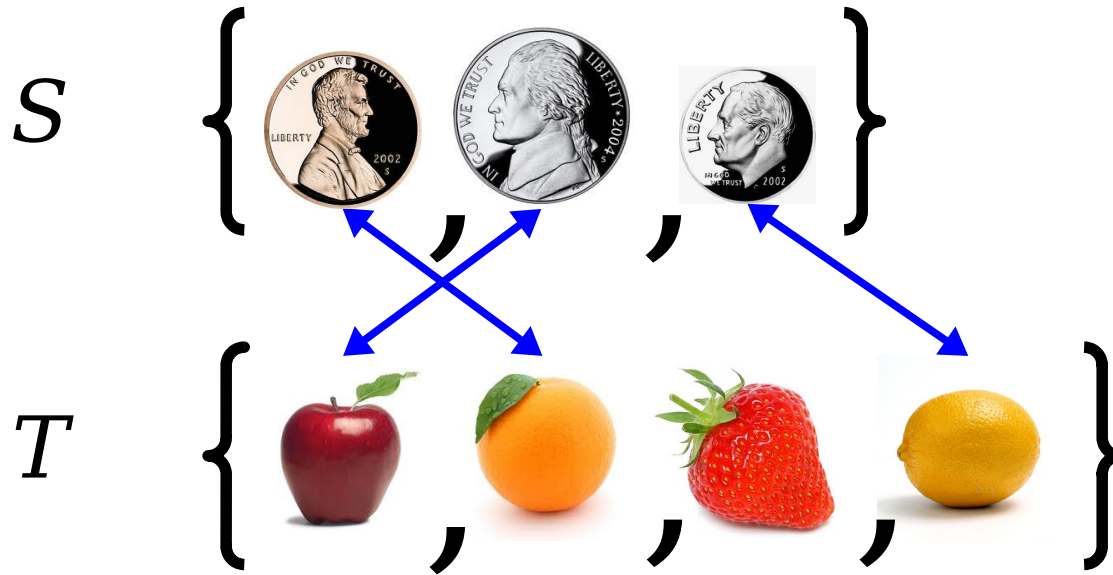
... say, these two sets here.





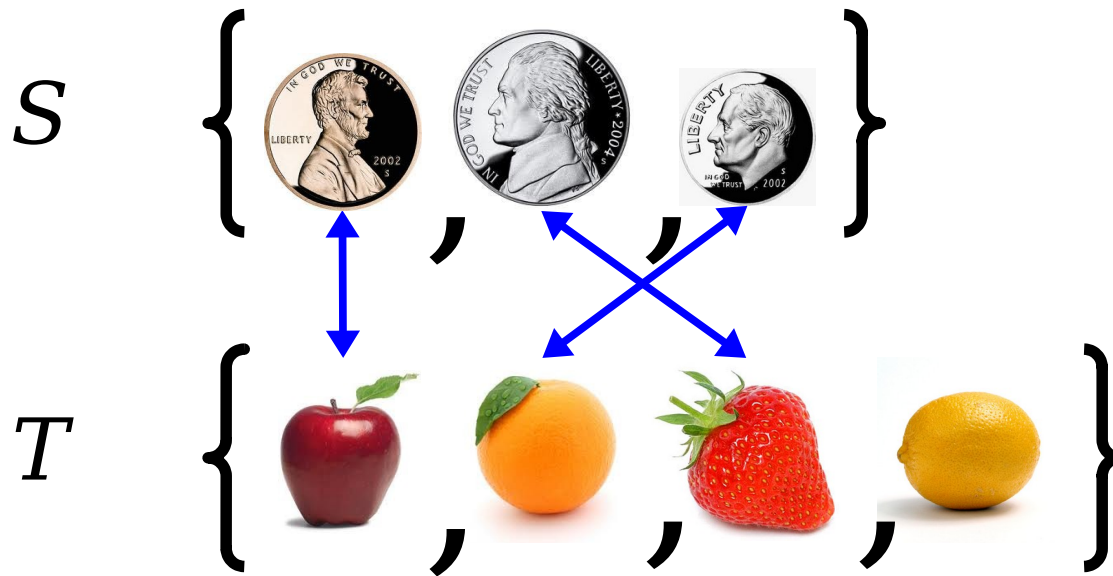
If we try pairing off the elements of S
and the elements of T ...





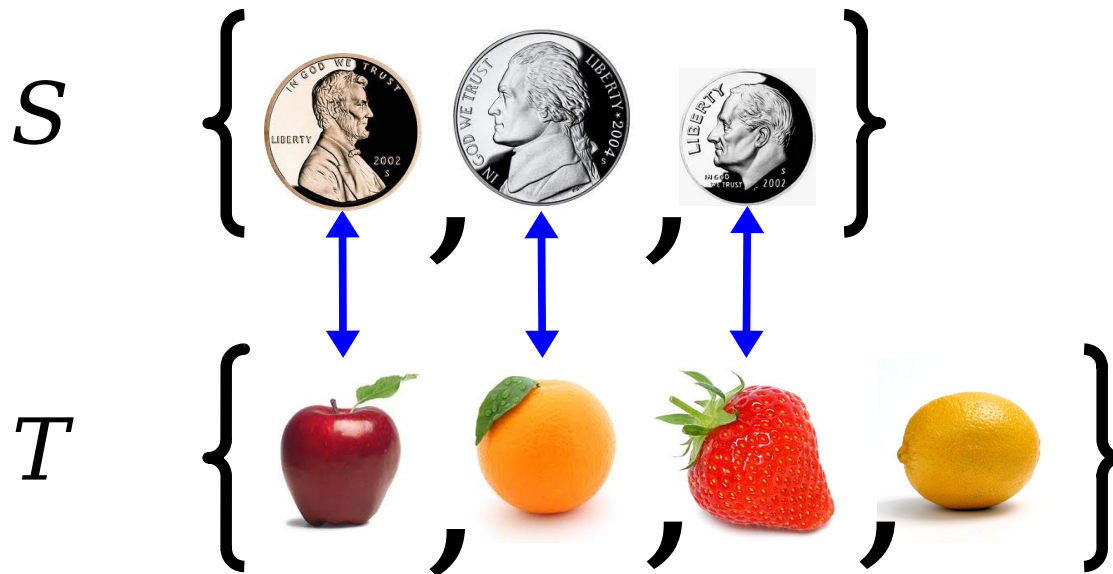
...we can see that there's some way to do it that uses up everything from S .



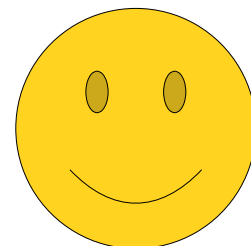


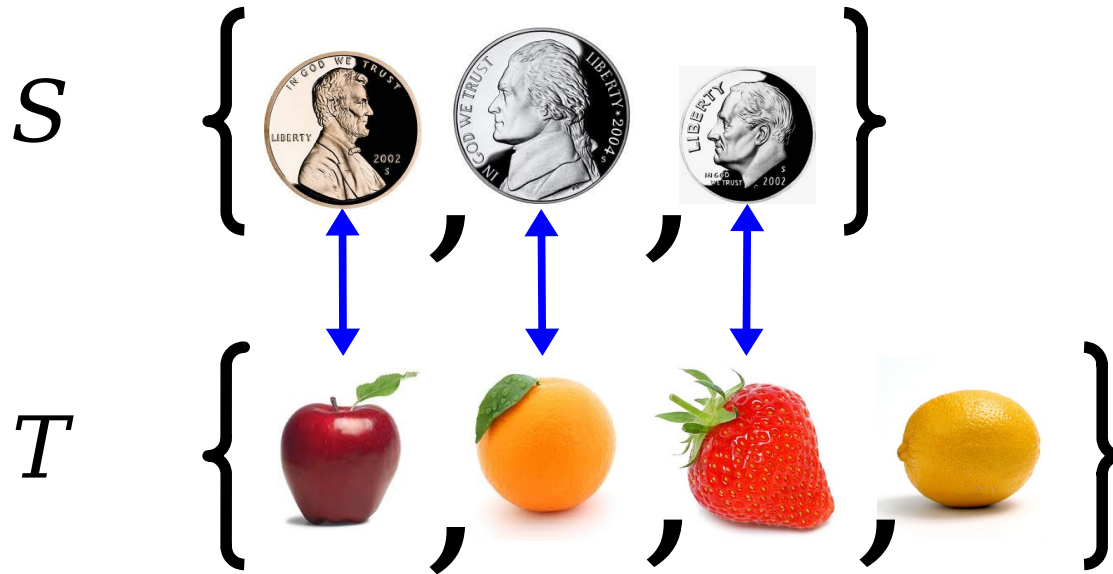
There's actually a bunch of ways to do this. Here's a different one...





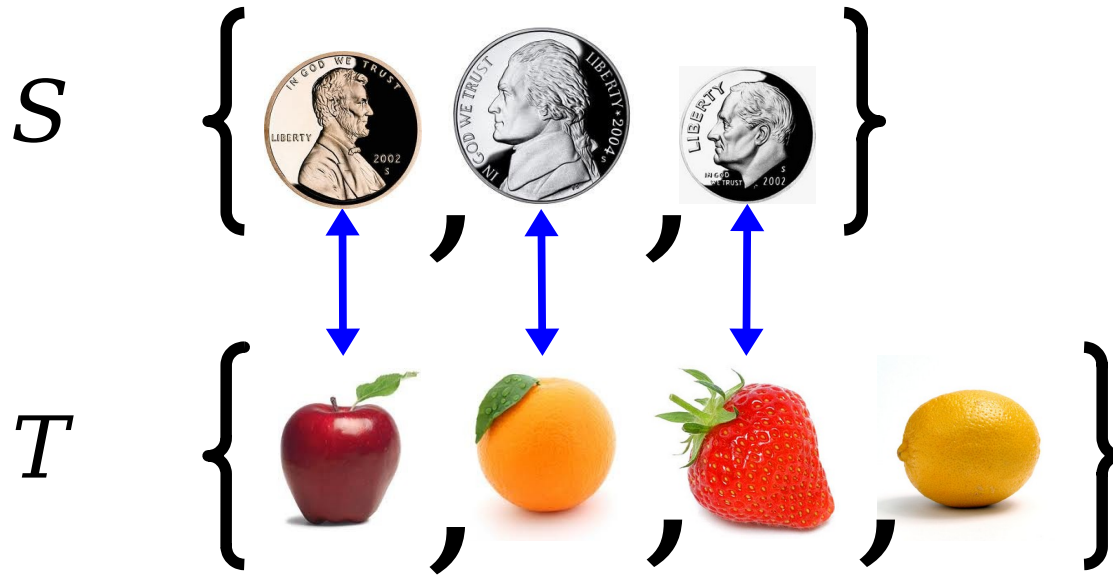
...and here's one more.





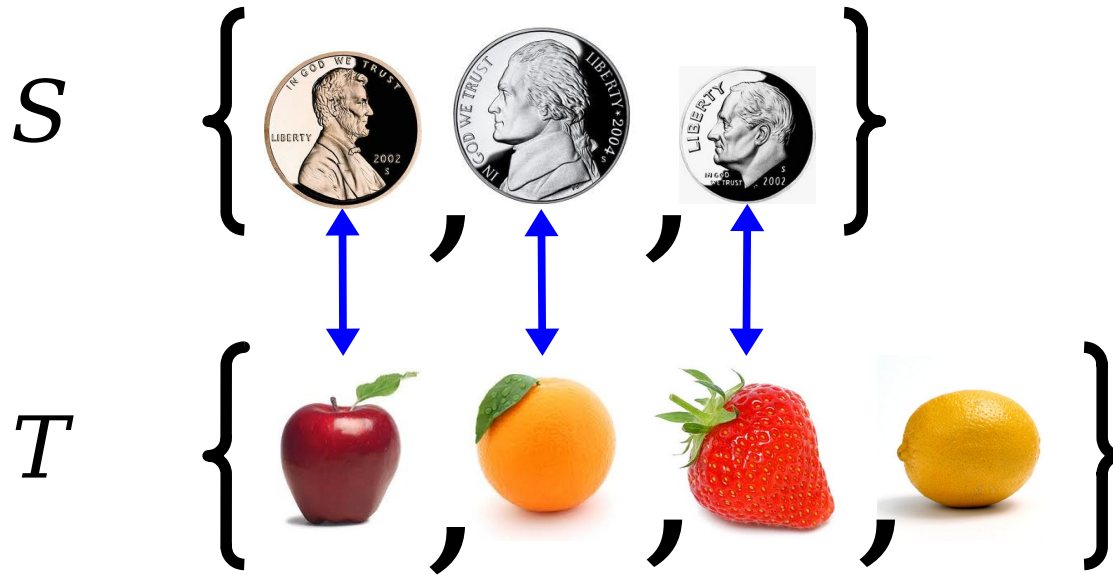
This gives us a nice intuition for what it means for one set to have at least as many elements as the other.





Specifically, we'll say that T is at least as big as S if there's a way to pair the elements that uses up everything in S , even if it doesn't use up everything in T .





Before moving on, let's make sure that this definition works in some other cases. If it doesn't, then chances are it's not a very good definition!



\mathbb{N}

0 1 2 3 ...

\mathbb{Z}

... -3 -2 -1 0 1 2 3 ...

For example, let's take a look at the
sets \mathbb{N} and \mathbb{Z} .



\mathbb{N}

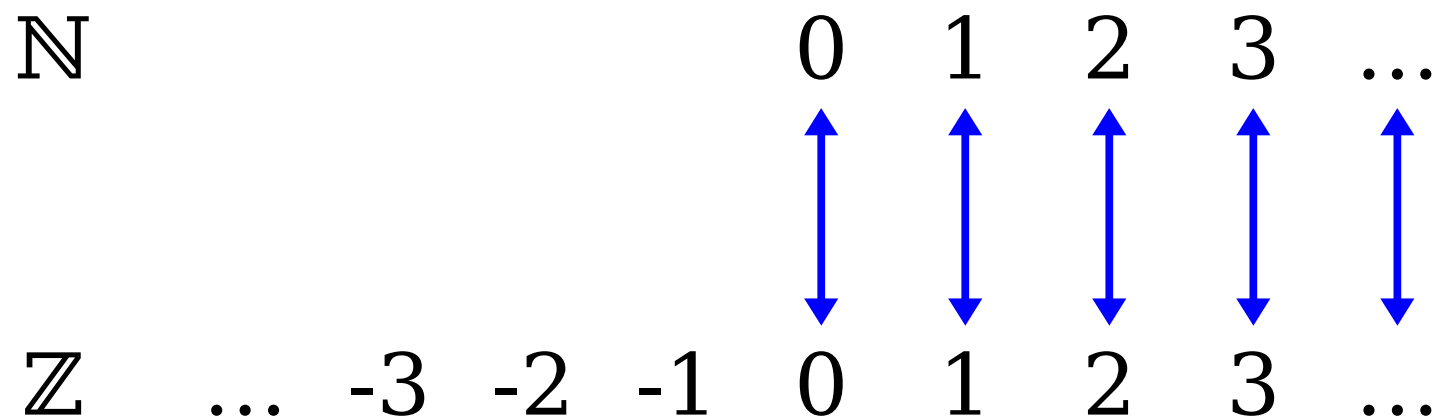
0 1 2 3 ...

\mathbb{Z}

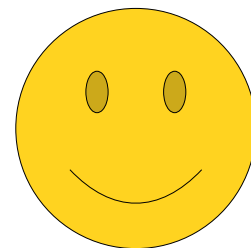
... -3 -2 -1 0 1 2 3 ...

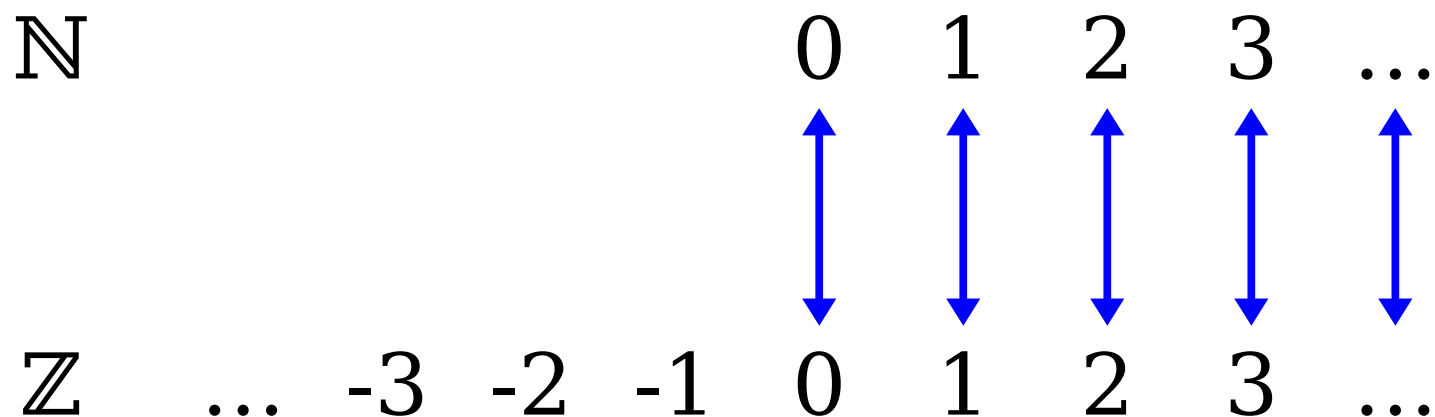
Can we pair off the naturals and the integers so that every natural number is paired off with some integer?



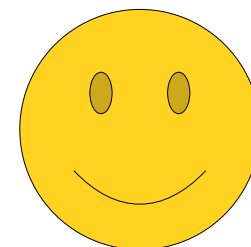


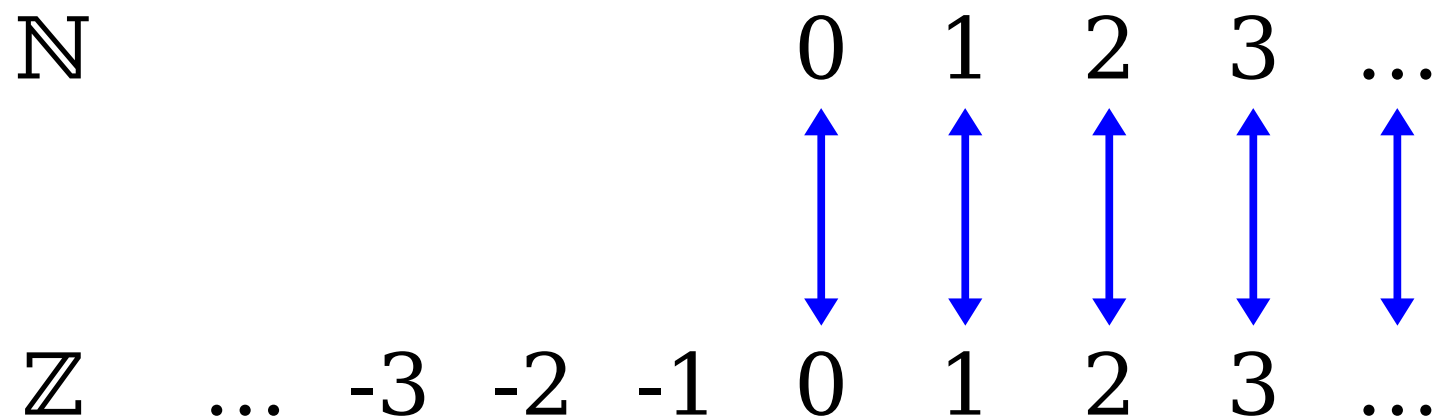
Yes!
Here's one very natural way to do this.



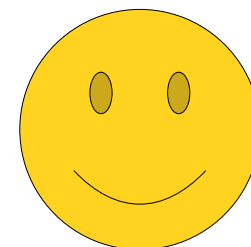


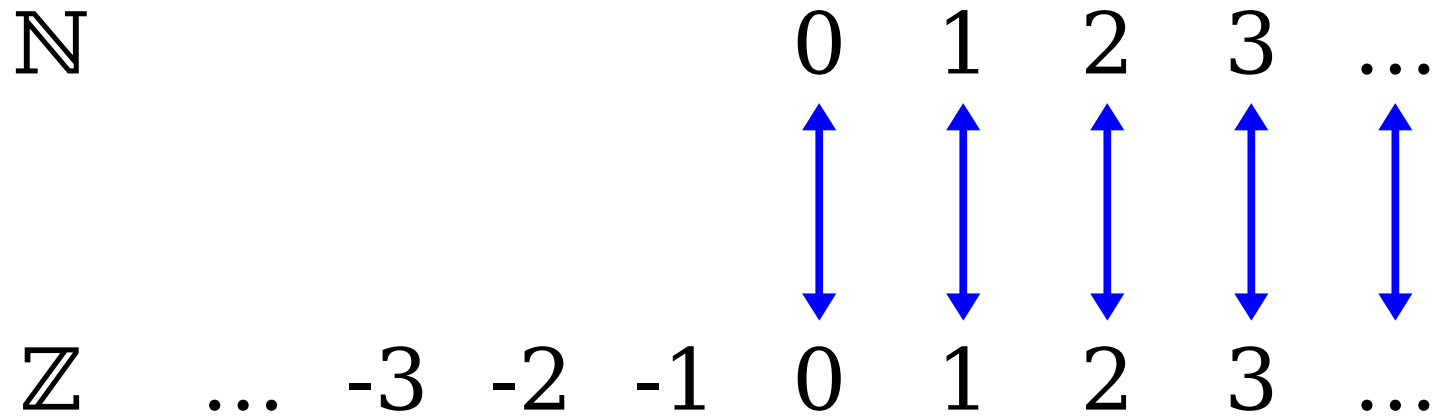
In lecture, we proved that $|\mathbb{N}| = |\mathbb{Z}|$.
 That means that we could in principle pair all the elements of both sets off, though it's a lot easier to do that if we don't need to pair off all the integers.





So far, it seems like this is a pretty good definition to work with!

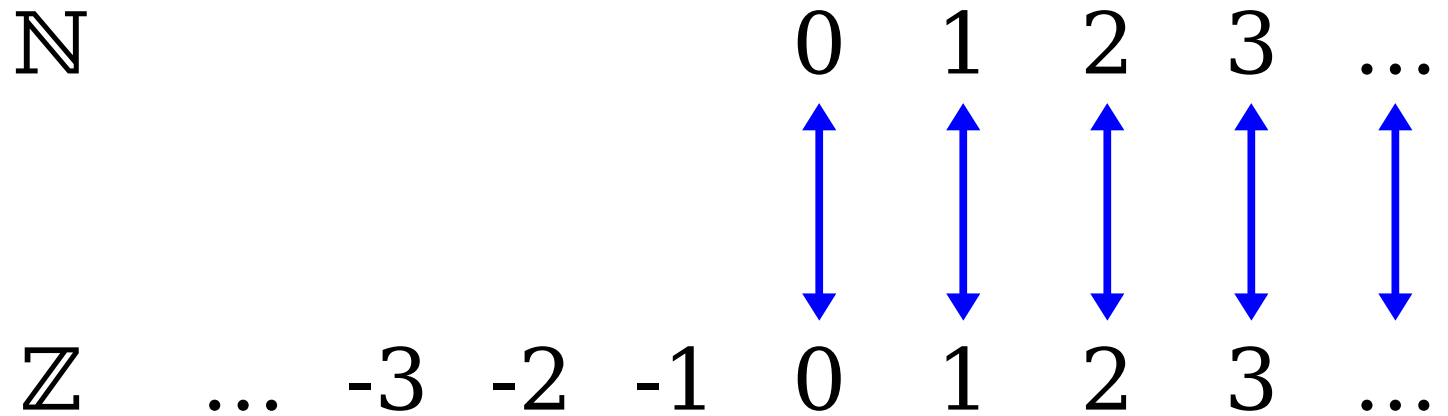




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

Here's what we have so far. This definition is expressed in terms of a pairing between the two sets.

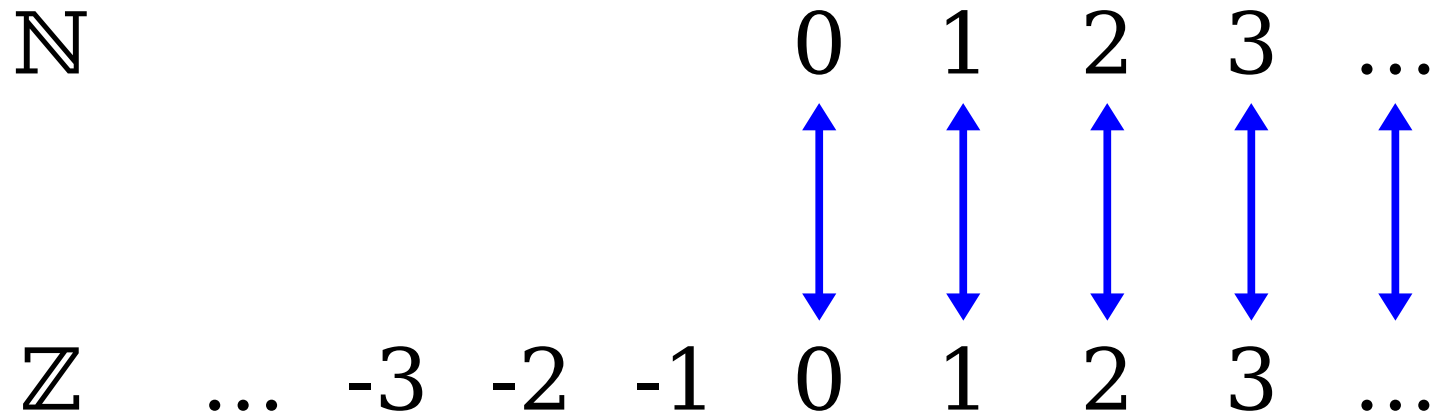




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

When we talked about equal cardinalities, we started with a definition like this one, then revised it by talking about the pairing in terms of a certain type of function between the two sets.

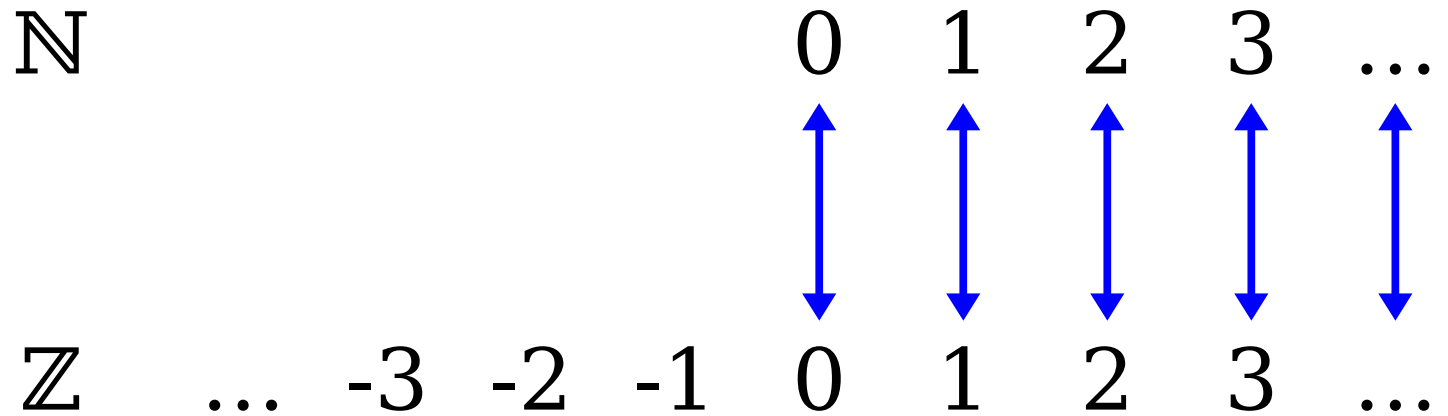




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

Could we do something like that here?

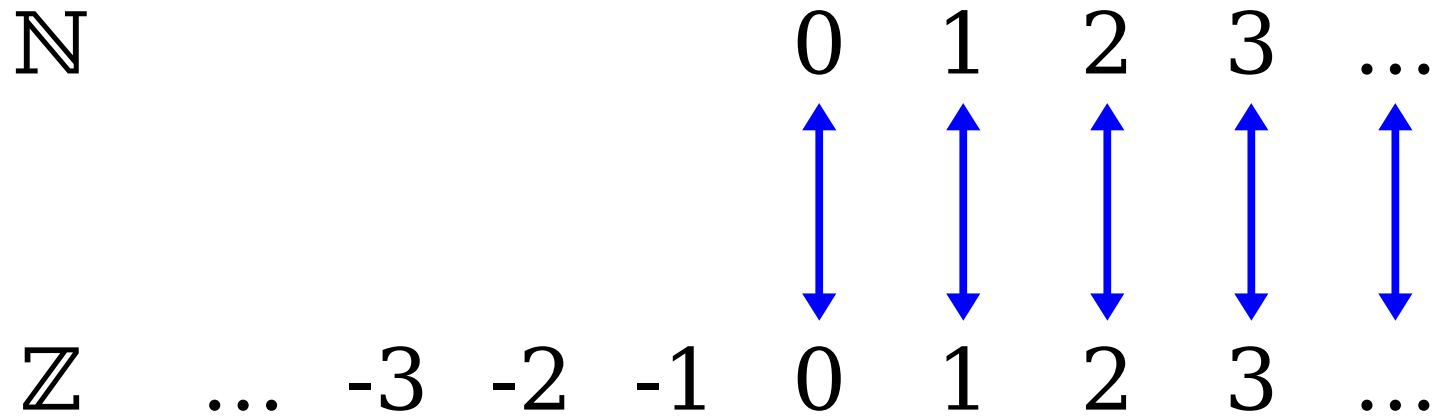




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

The answer is yes!

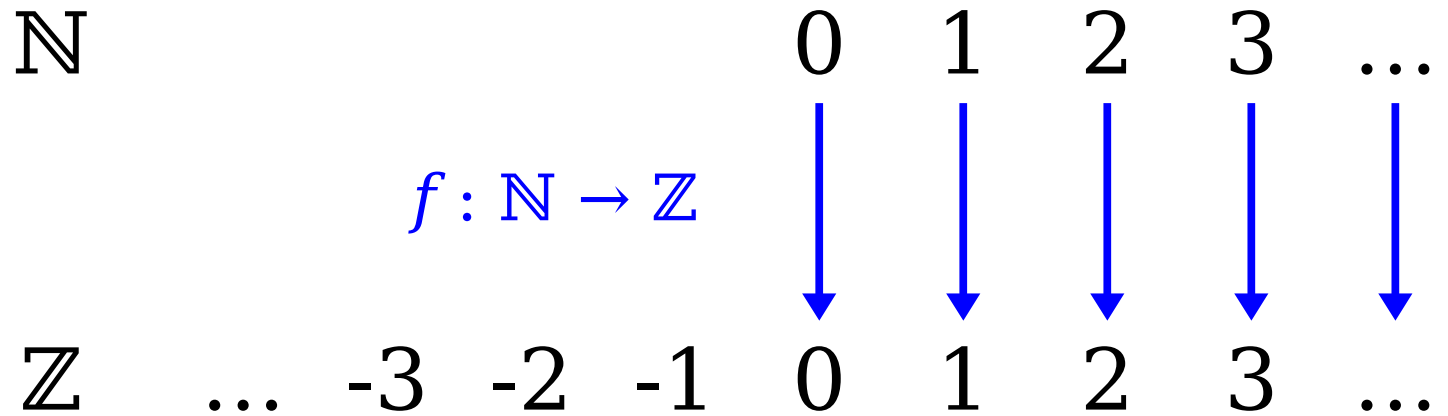




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

Let's look at the pairing we have right here.

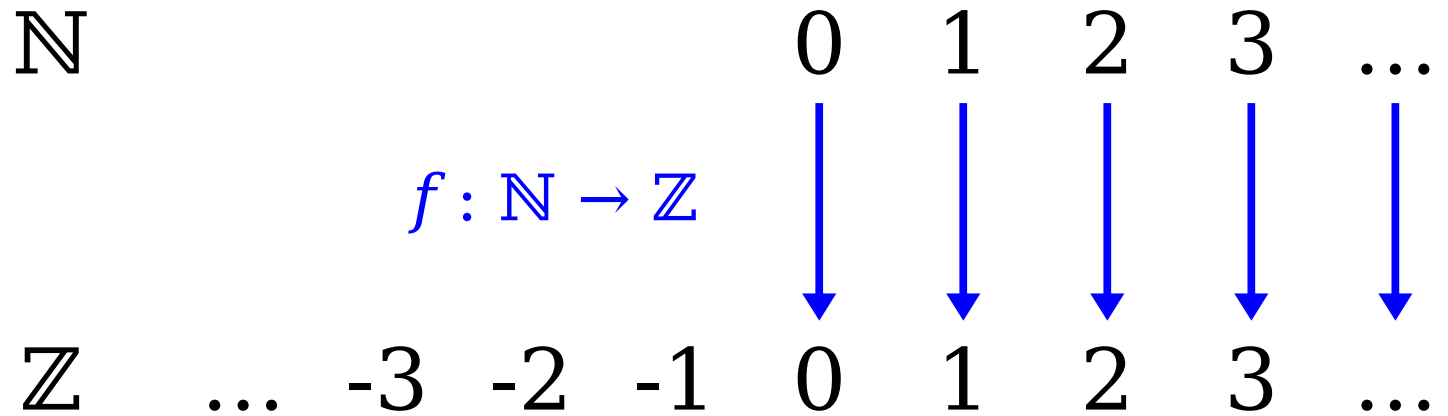




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

Let's imagine it as a function from \mathbb{N} to \mathbb{Z} .

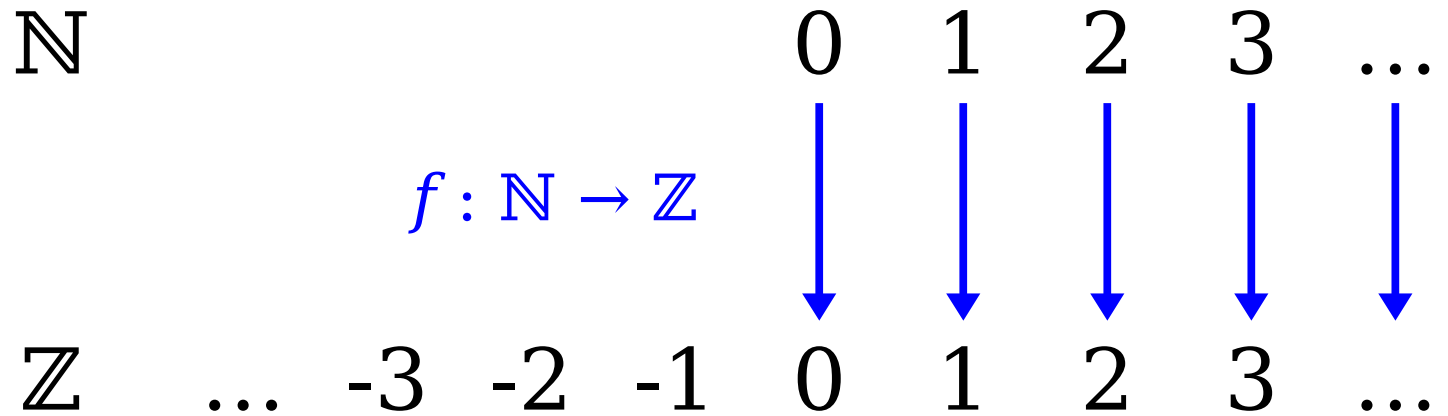




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

What properties will this function have?

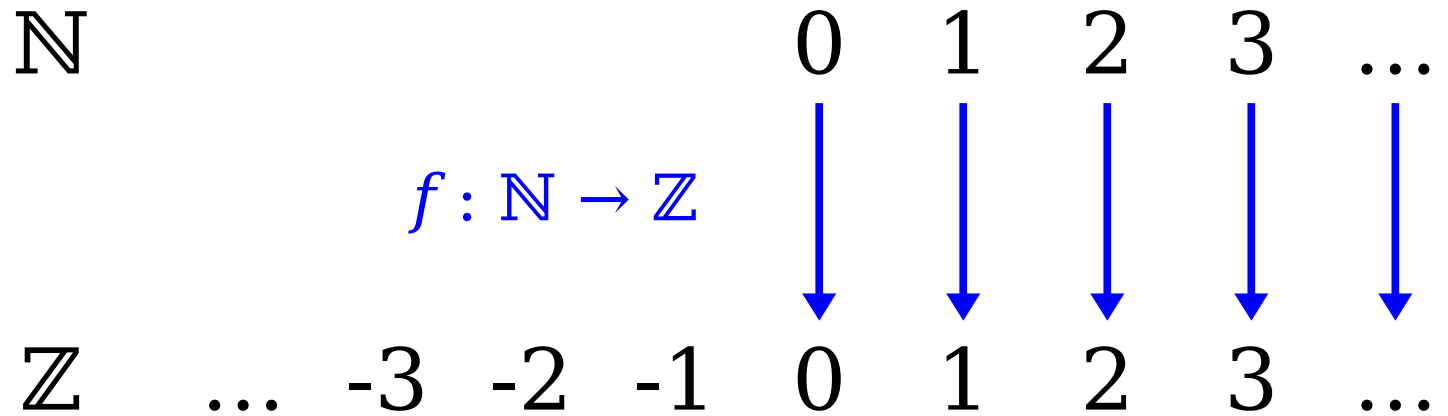




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

Well, since it's pairing off elements of \mathbb{N} with elements of \mathbb{Z} , we know that different elements of \mathbb{N} map to different elements of \mathbb{Z} .





$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

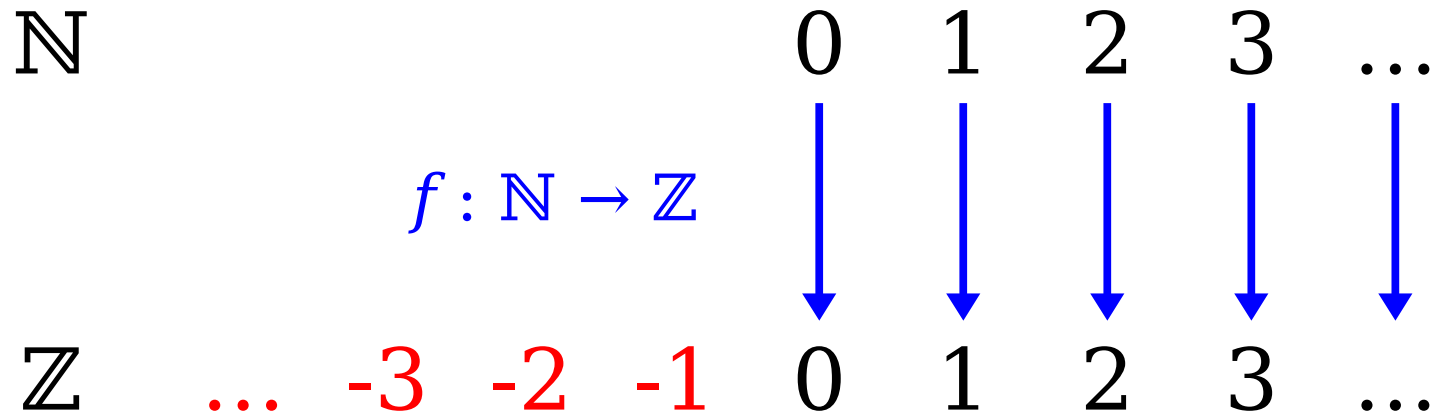
That's the definition of an injective function!





However, this particular function is not surjective, since it leaves a lot of elements of \mathbb{Z} uncovered.





$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

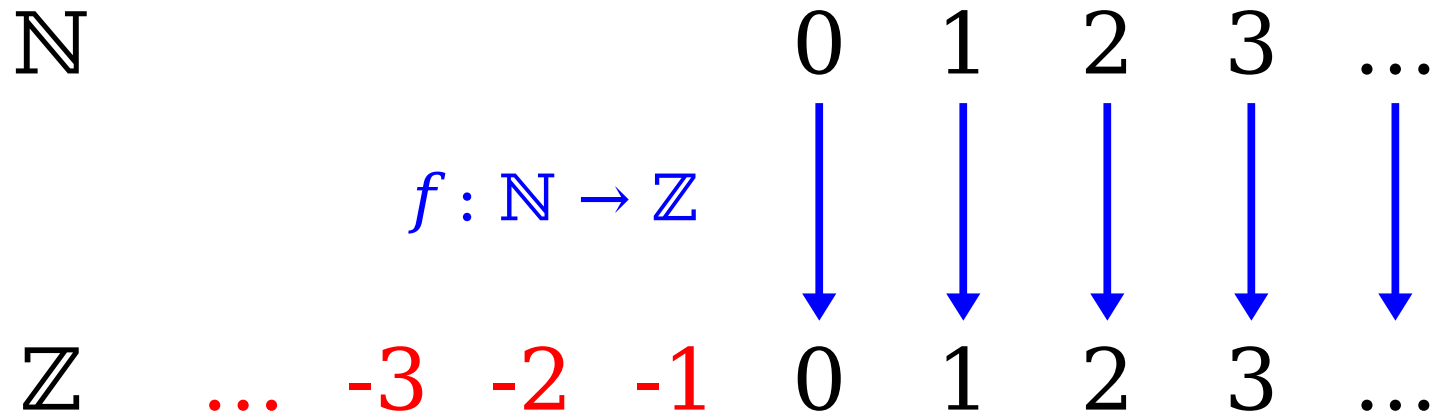
That means that this function isn't a bijection, but it is injective.





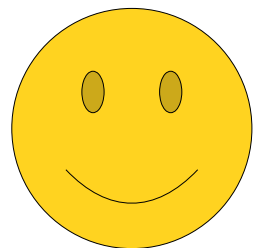
So it seems like our idea of "a pairing of elements that uses everything from the first set, but not necessarily the second" can be expressed as "an injective function from the first set to the second!"

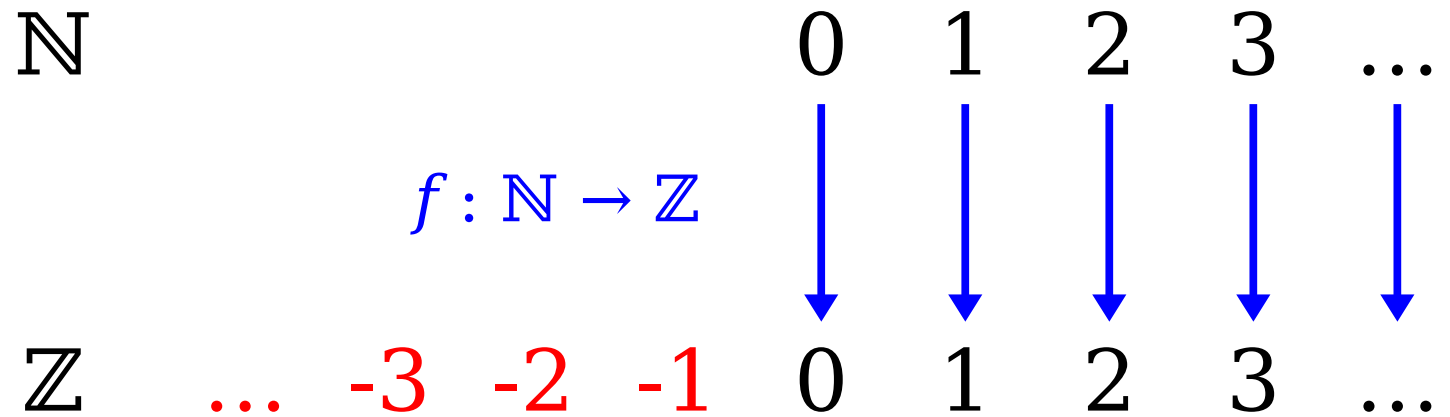




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S , but (not necessarily) all the elements of T .

In fact, that's precisely how we're going to define the \leq relation over cardinality.





$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

Here's the official definition of what it means for one set to be at least as large as another.



$|S| \leq |T|$ if there is an **injective** function $f : S \rightarrow T$

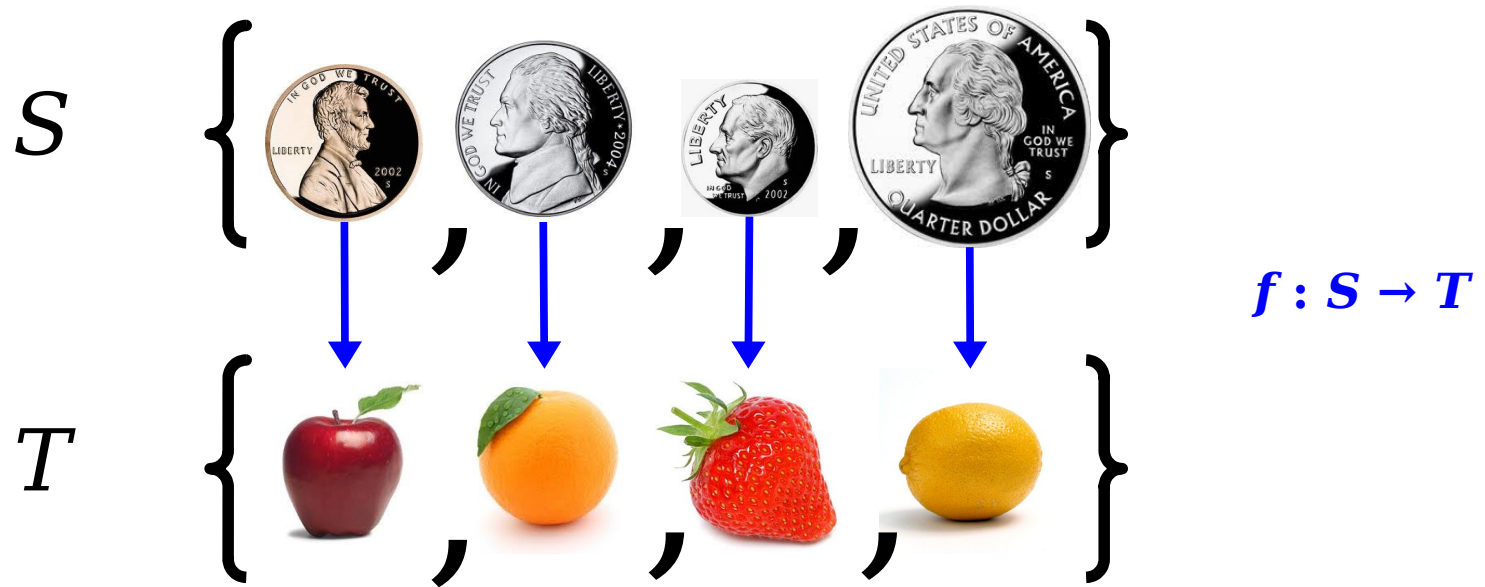
You might be wondering... why is this the definition of "less than or equal to" for sets rather than "strictly less than?"



$|S| \leq |T|$ if there is an ***injective*** function $f : S \rightarrow T$

To see why this is, let's look at an example.

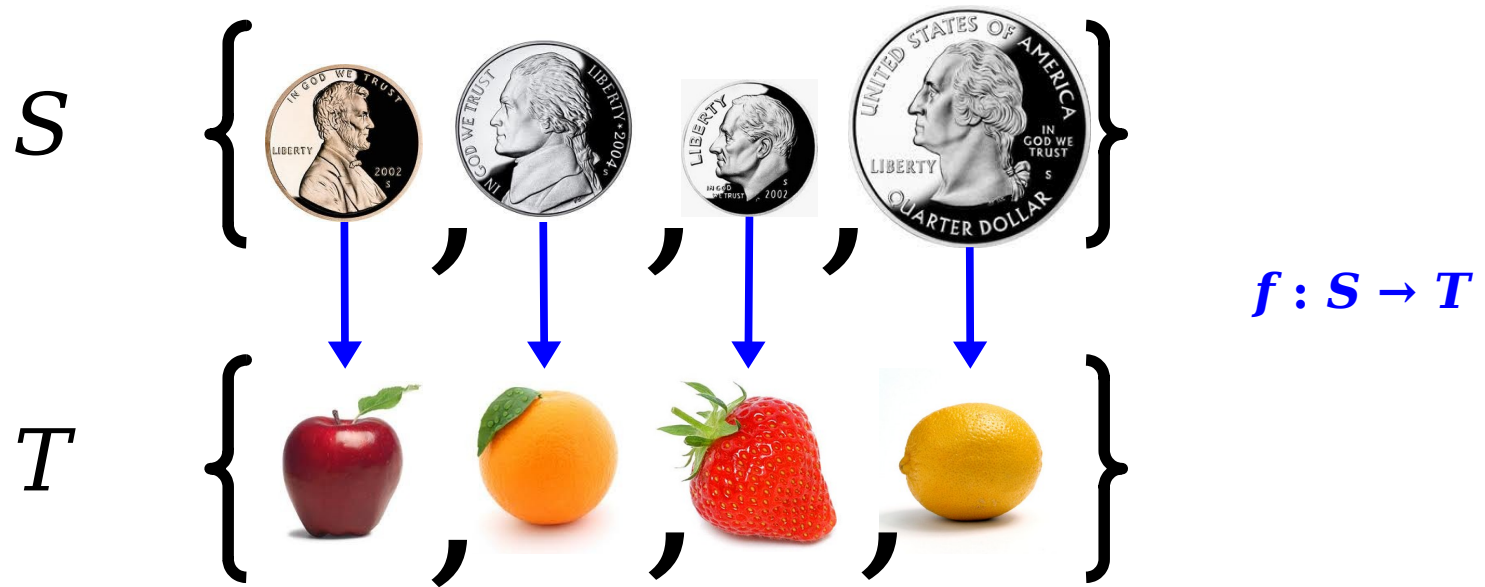




$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

Here's two sets we've seen from before,
along with a bijection between them.

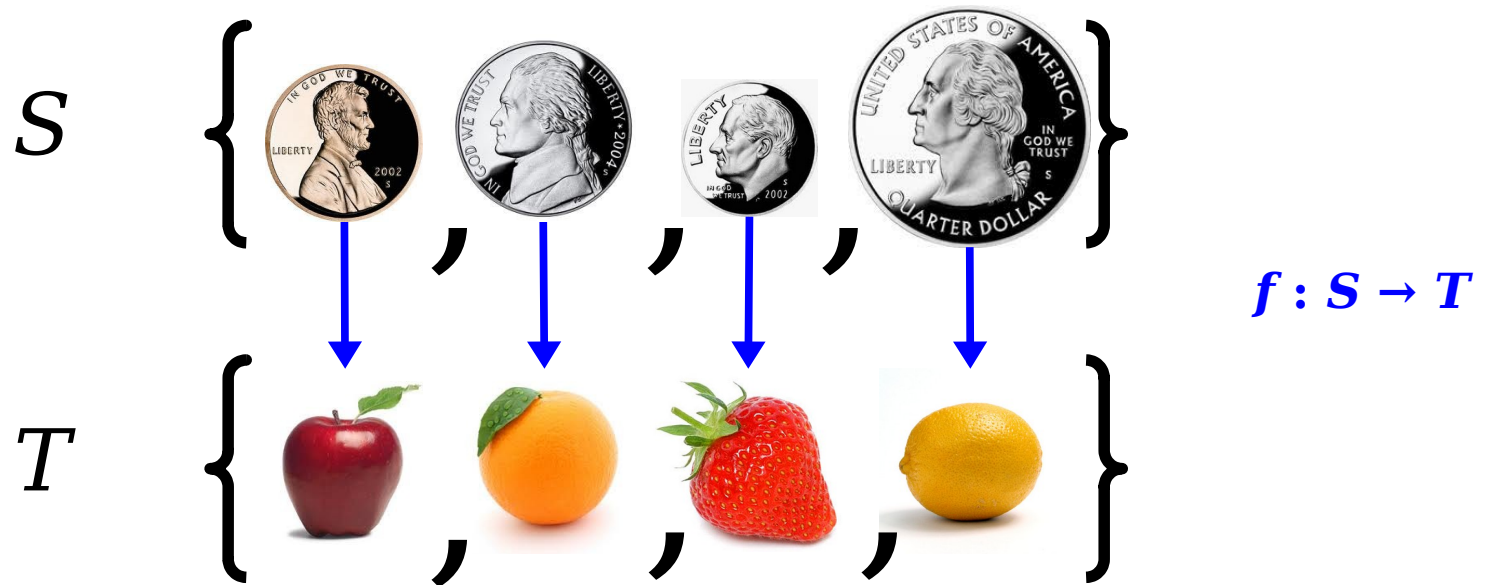




$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

Because f is a bijection from S to T ,
we know that $|S| = |T|$.

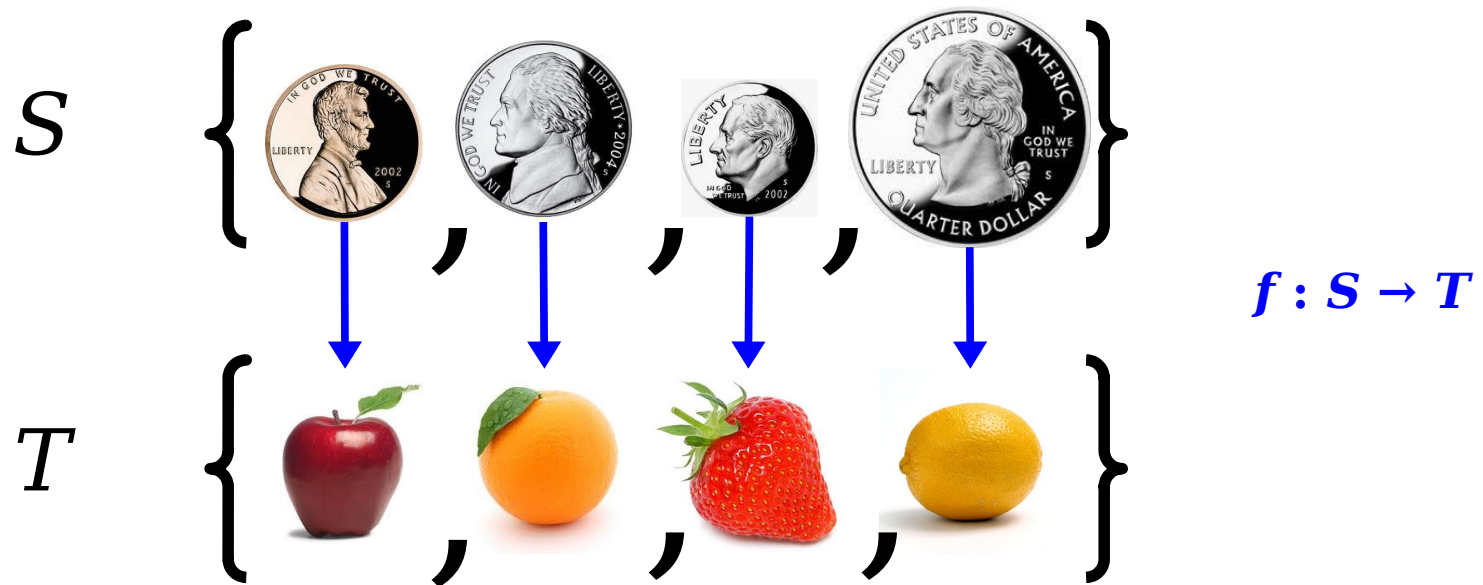




$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

At the same time, this function f is also an injection (remember, all bijections are also injective!)

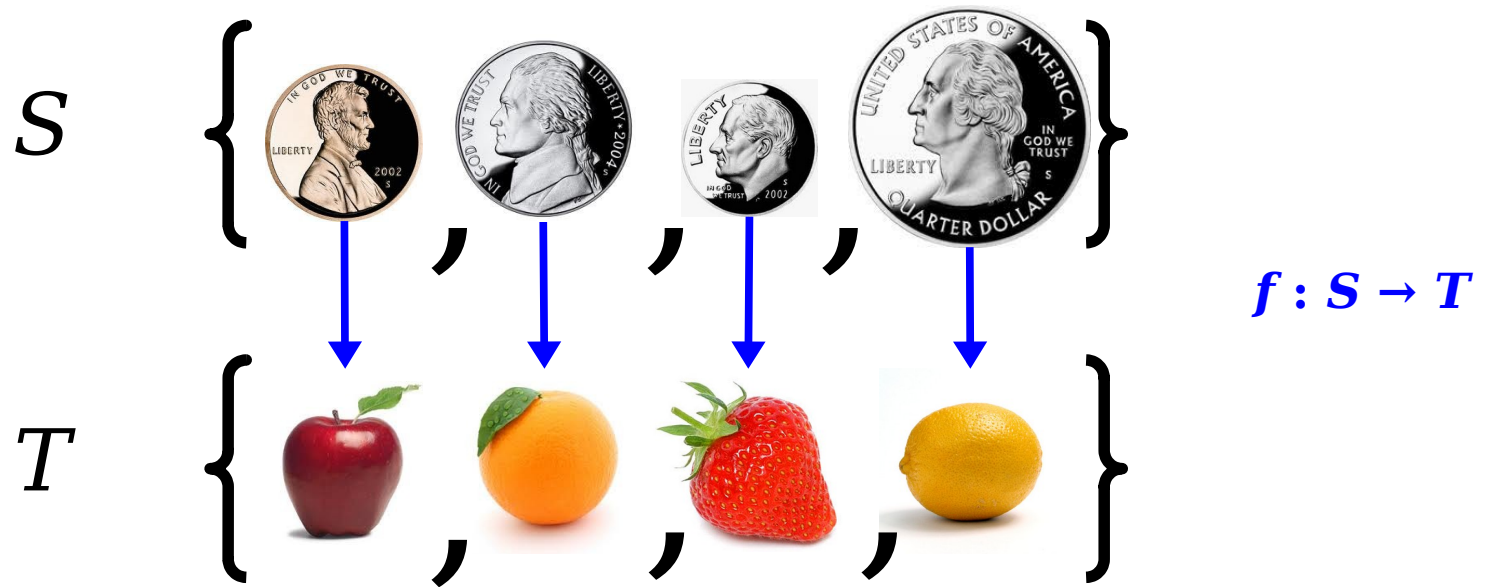




$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

So, according to our definition, this means that $|S| \leq |T|$. That's okay, though - it matches our intuition.





$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

That's why we use this as a definition of \leq over cardinalities – it's to be consistent with our definition of $=$ over cardinalities.



If S is a set, then $|S| < |\wp(S)|$

so let's jump back to the statement of
Cantor's theorem.



If S is a set, then $|S| < |\wp(S)|$

Earlier, we said that we couldn't proceed to prove this statement because we didn't have a formal definition for this statement.



If S is a set, then $|S| < |\wp(S)|$

We just found a way to define the \leq relation over set cardinalities, but we still haven't said what the $<$ relation over cardinalities is!



If S is a set, then $|S| < |\wp(S)|$

In other words, we still can't prove this
because we still don't have a definition!



If S is a set, then $|S| < |\wp(S)|$

So why did we go down what seems like
it was probably a totally random detour?



If S is a set, then $|S| < |\wp(S)|$

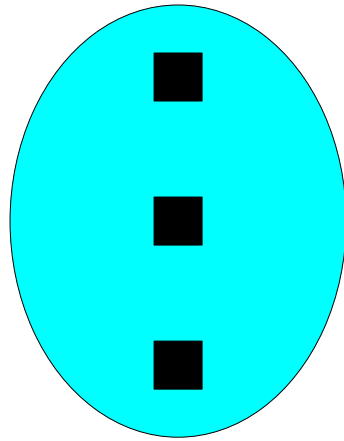
The reason is that we now have formal definitions for $|S| \leq |T|$ and $|S| = |T|$.



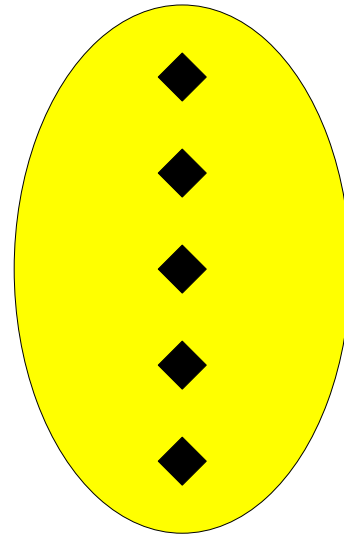
If S is a set, then $|S| < |\wp(S)|$

Can we use those definitions to provide
a nice definition of what $|S| < |T|$ means?





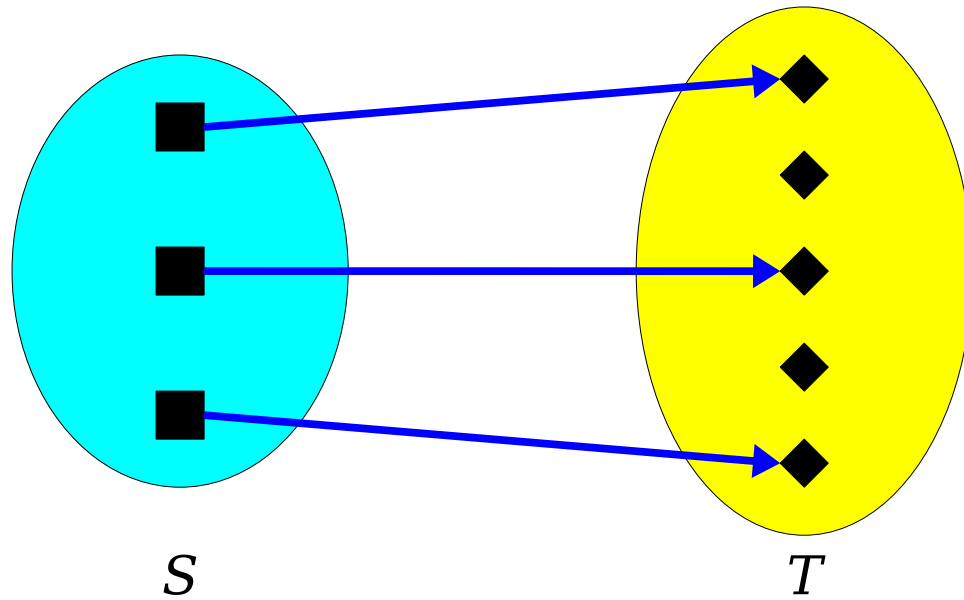
S



T

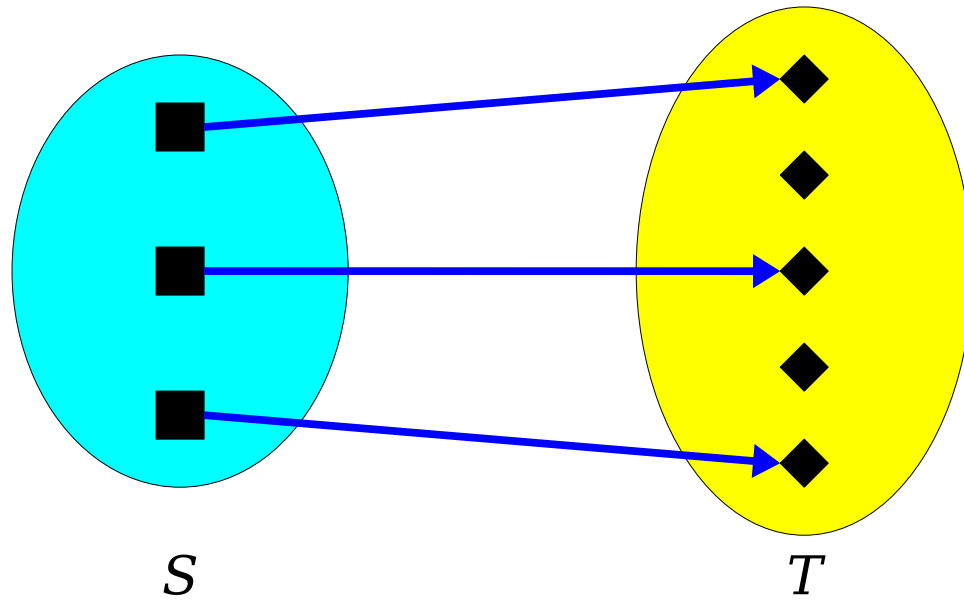
Let's take a look at these two sets
 S and T .





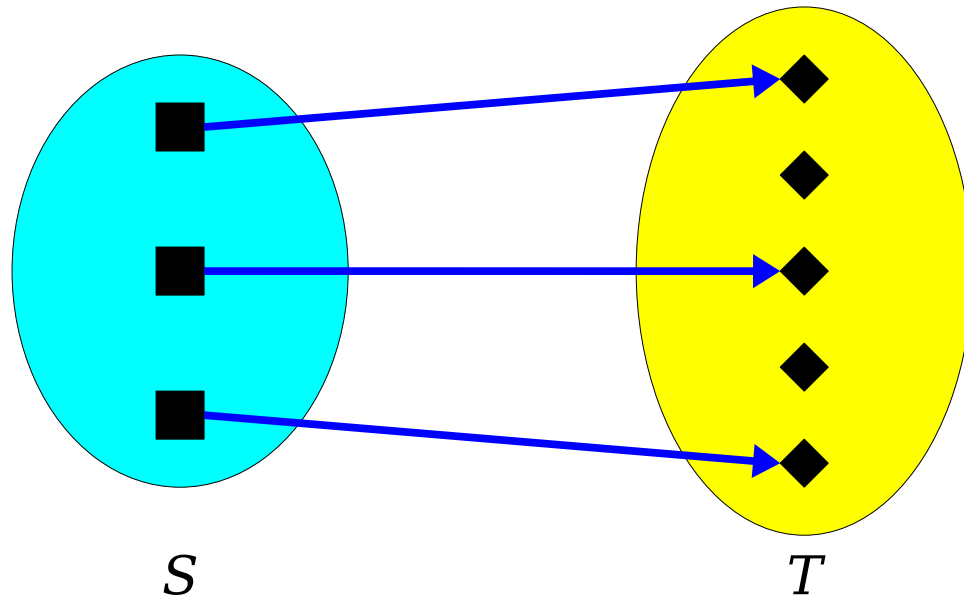
First, notice that $|S| \leq |T|$, since we can find an injection from S to T .





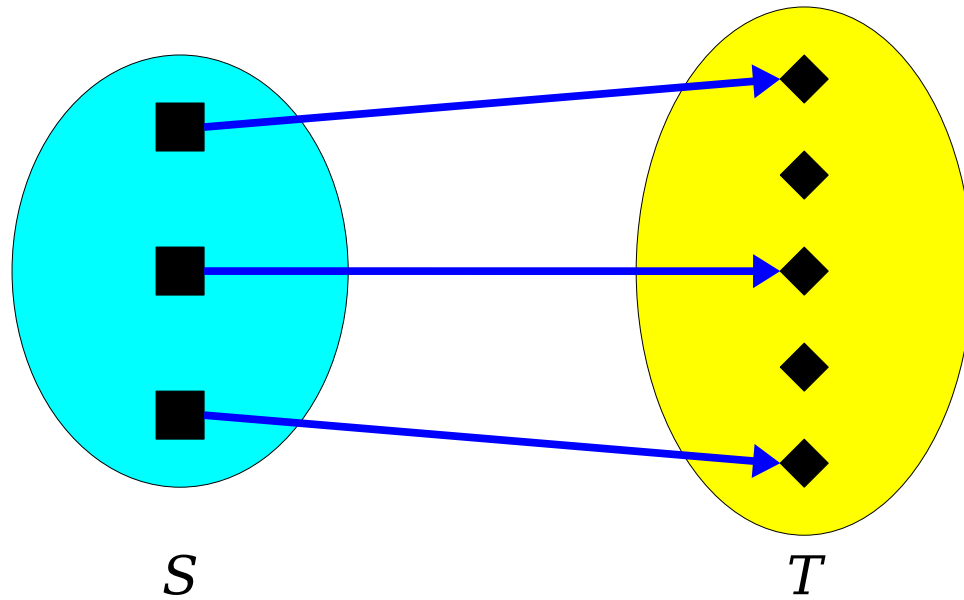
We can also see that $|S| \neq |T|$, since no matter how hard we try, we'll never find a bijection between the two sets.





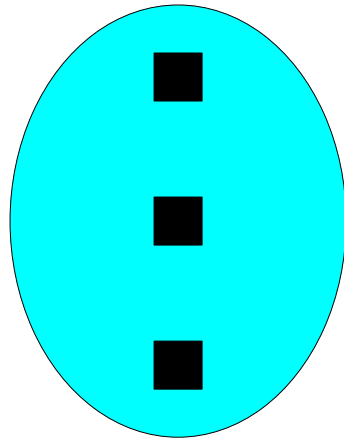
So we've seen that $|S| \leq |T|$ and
that $|S| \neq |T|$...



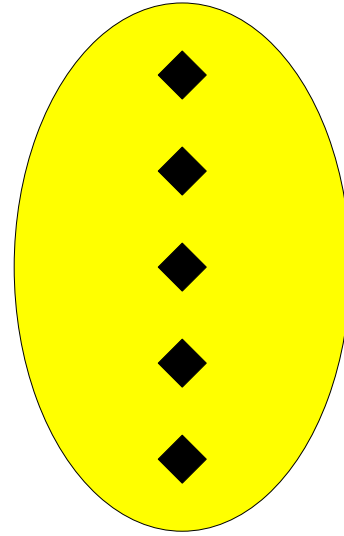


...and intuitively it seems like the set S
is strictly smaller than the set T .





S



T

Perhaps that observation would make for
a good definition of less-than over
cardinalities!



$|S| < |T|$ **if** $|S| \leq |T|$ **and** $|S| \neq |T|$

In fact, here's how we're going to
define the less-than relation over
set cardinalities.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

This seems, intuitively, like it's a pretty nice definition. It makes sense given our understanding of what \leq and $=$ are supposed to mean and how they relate to $<$.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

But remember that the \leq and \neq relations over cardinalities are defined in terms of functions between sets. Let's see what this definition says when we expand this out a bit.



$$|S| < |T| \quad \text{if} \quad \underbrace{|S| \leq |T| \text{ and } |S| \neq |T|}$$

There is an injective
function $f: S \rightarrow T$

This part of the definition says that
there should be an injective function
from S to T ...



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

...and this part says that there aren't
any bijective functions from S to T .



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

So, if we want to prove that one set is
strictly smaller than one another, we
should proceed in two parts.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

First, we should find an injective
function from the smaller set to the
larger one.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

Second, we should prove that there are
no bijections from the smaller set to
the larger one.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

If we can prove both of these statements,
then we've shown everything that we've
needed to show!



If S is a set, then $|S| < |\wp(S)|$

so let's now take a deeper look at
Cantor's theorem.



If S is a set, then $|S| < |\wp(S)|$

We now have a formal definition for
this statement!



There is an injection $f : S \rightarrow \wp(S)$
There is *no* bijection $f : S \rightarrow \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

We now know what we need to prove!
We need to find an injection from
 S to $\wp(S)$ and to prove that
there's no bijection from S to $\wp(S)$.



There *is* an injection $f : S \rightarrow \wp(S)$
There is *no* bijection $f : S \rightarrow \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

Let's start with this part. Turns out,
it's not too bad!



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

We now have this task set up before us.
How exactly might we go about doing
this?



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Well, when confronted with a new problem to solve, it never hurts to draw some pictures and try to figure out what it is that we need to do.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

First, let's refresh some terminology.

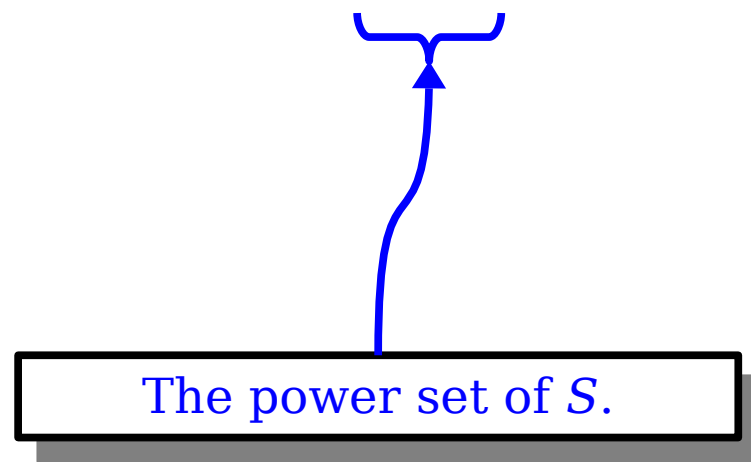


Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

What exactly is $\wp(S)$?



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



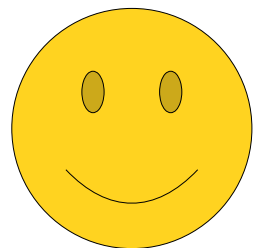
This is the power set of s .



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S .
The set of all subsets of S .

That's the set of all subsets of s .



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S .
The set of all subsets of S .

So if our goal is to find an injection from S to $\wp(S)$, we're looking for an injection from the set S to the set of all subsets of S .



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Let's start with a nice set S , say,
the set $\{1, 2, 3\}$.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



We'll draw S over here.



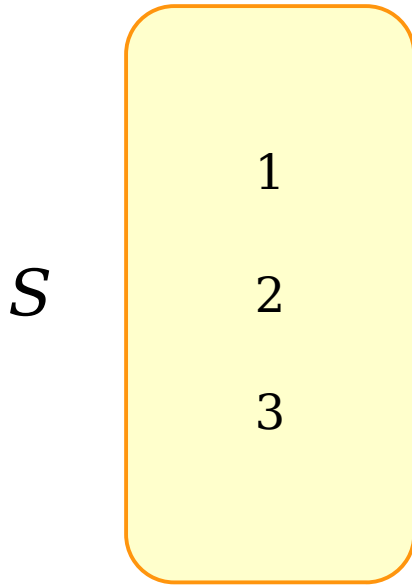
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Now, let's think about what the power set of S looks like.



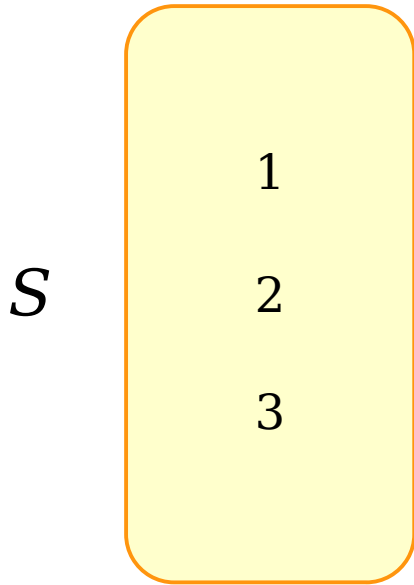
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



If you're not sure what elements are in that set, take a minute to work it out on your own.



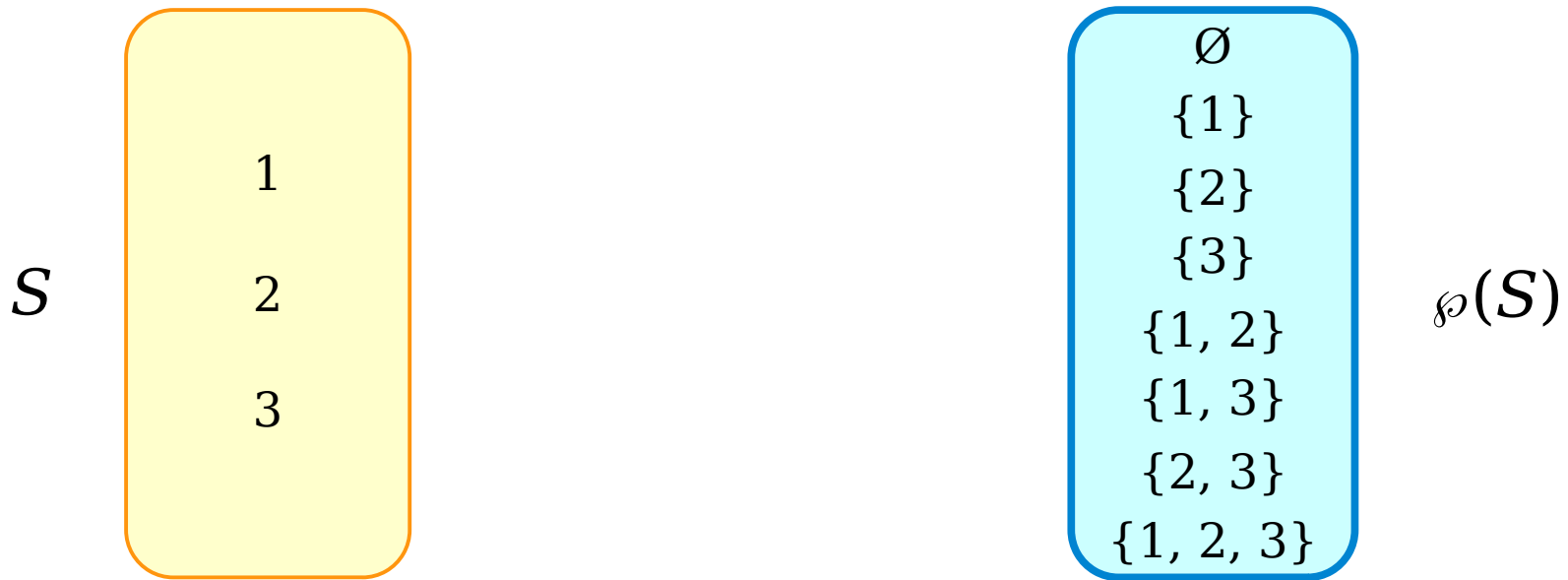
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



So you're good on what the power set contains? Awesome!



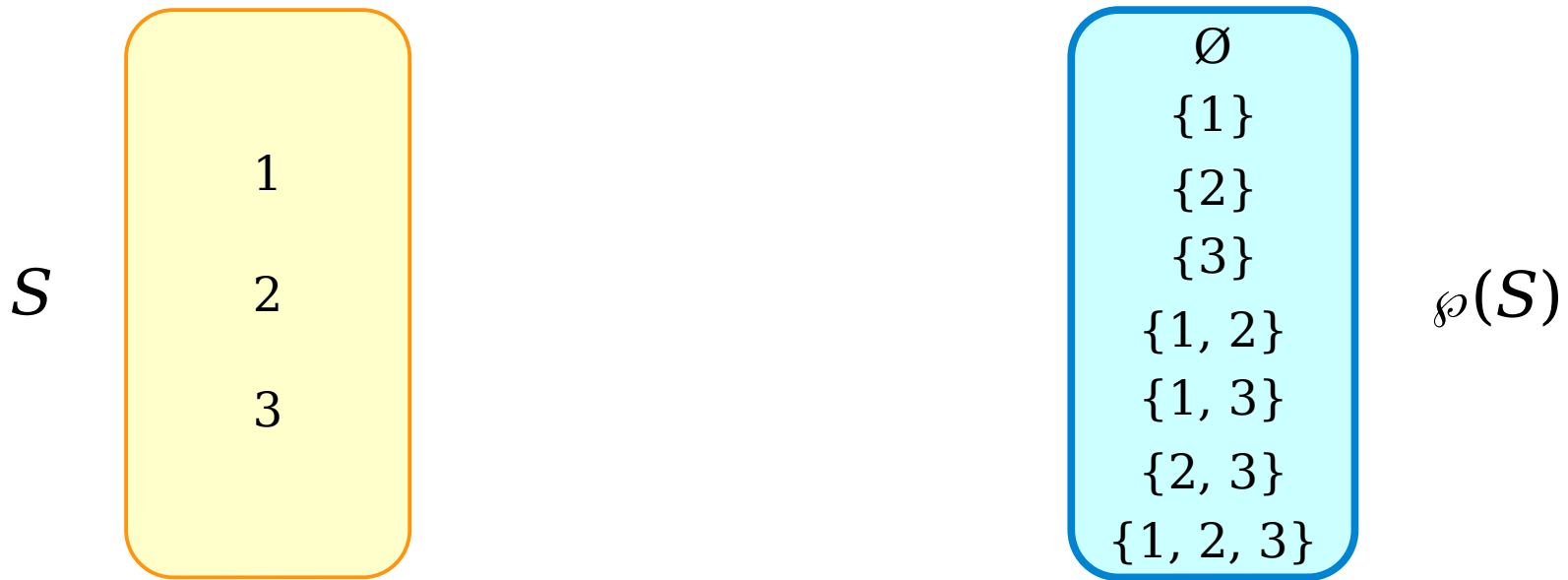
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



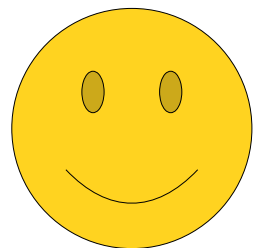
Here's what that looks like.



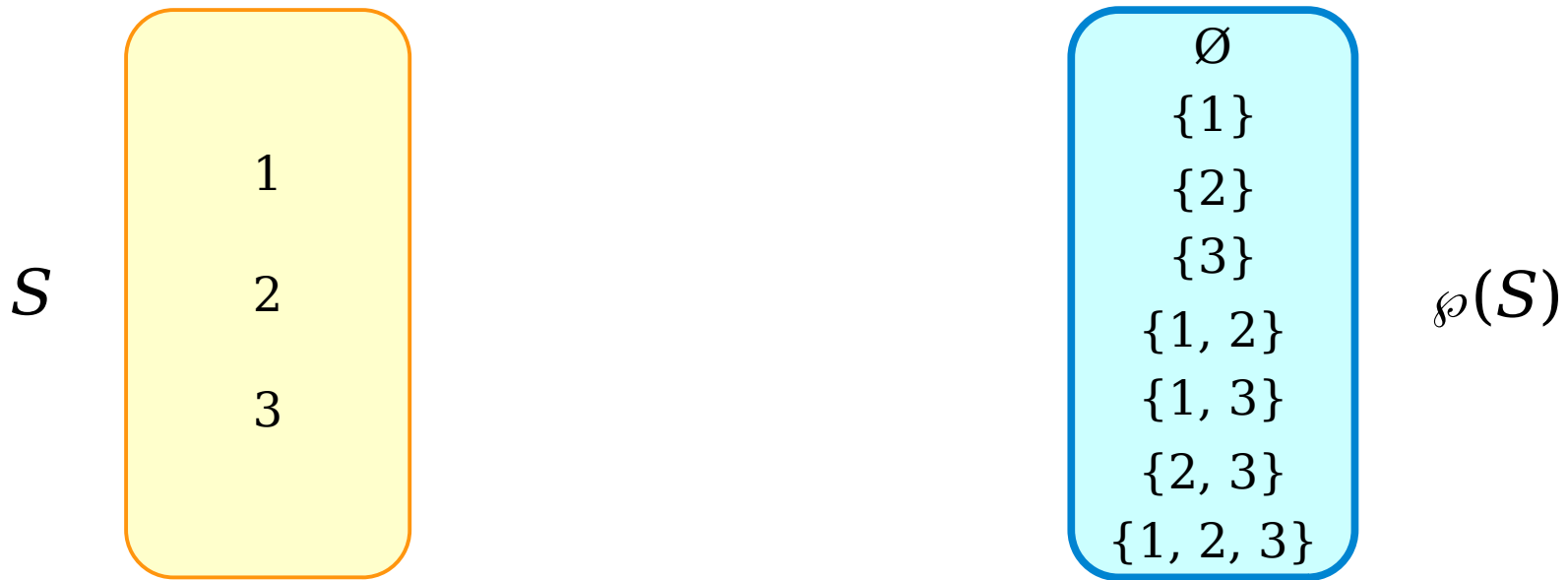
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Now, we need to find an injection here from S to $\wp(S)$.



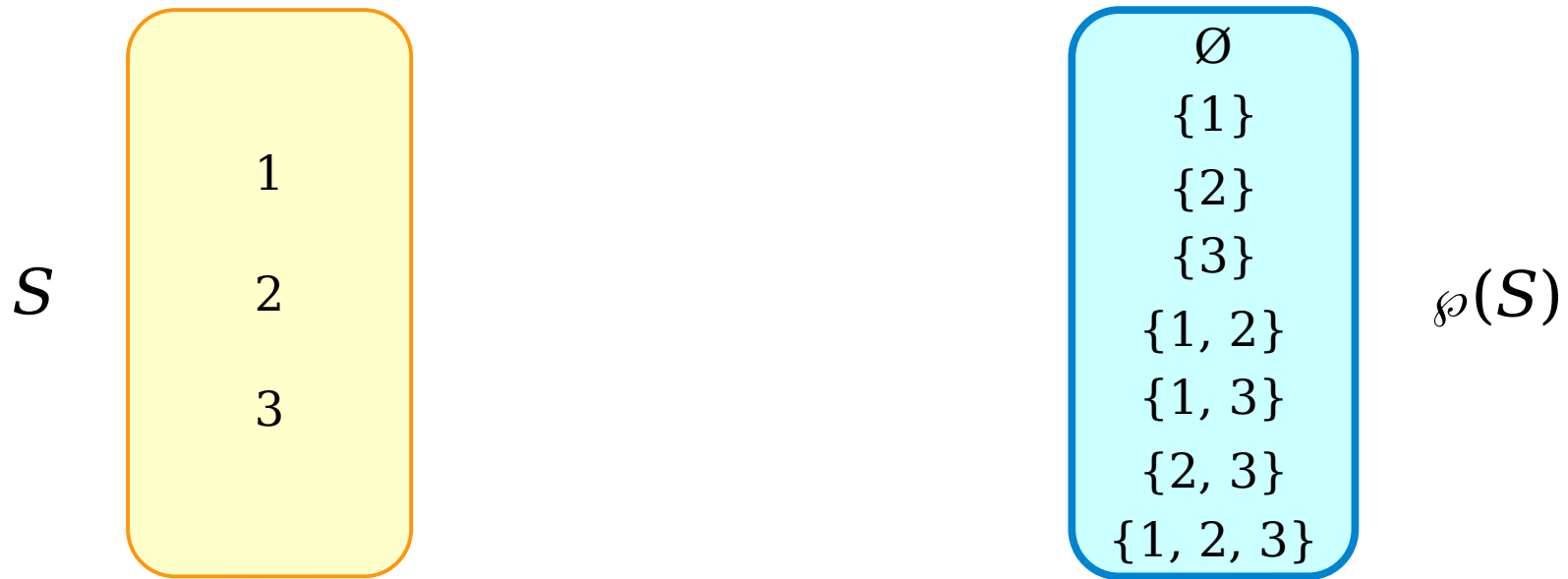
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Remember our intuition from before – we want to pair off the elements of S with the elements of $\wp(S)$ so that we use up all the elements of S .



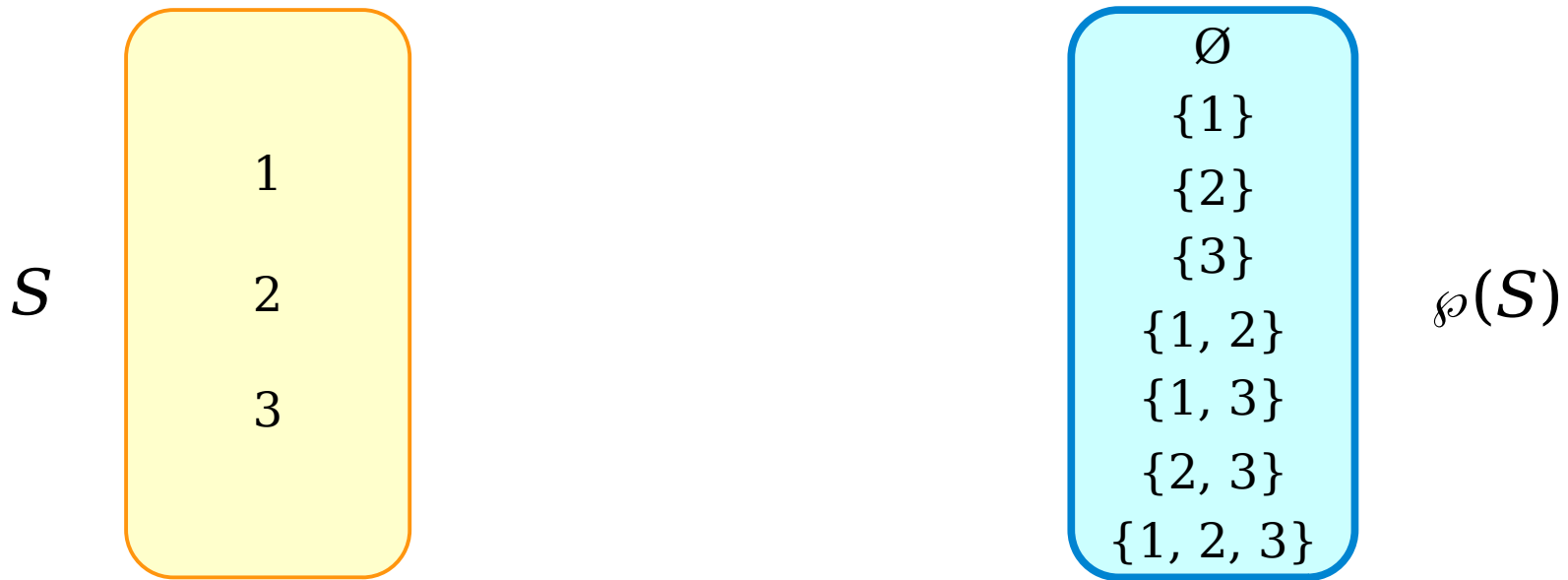
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



It's okay if some elements of $\wp(S)$ are left over – after all, we're ultimately trying to prove that $|S| < |\wp(S)|$!



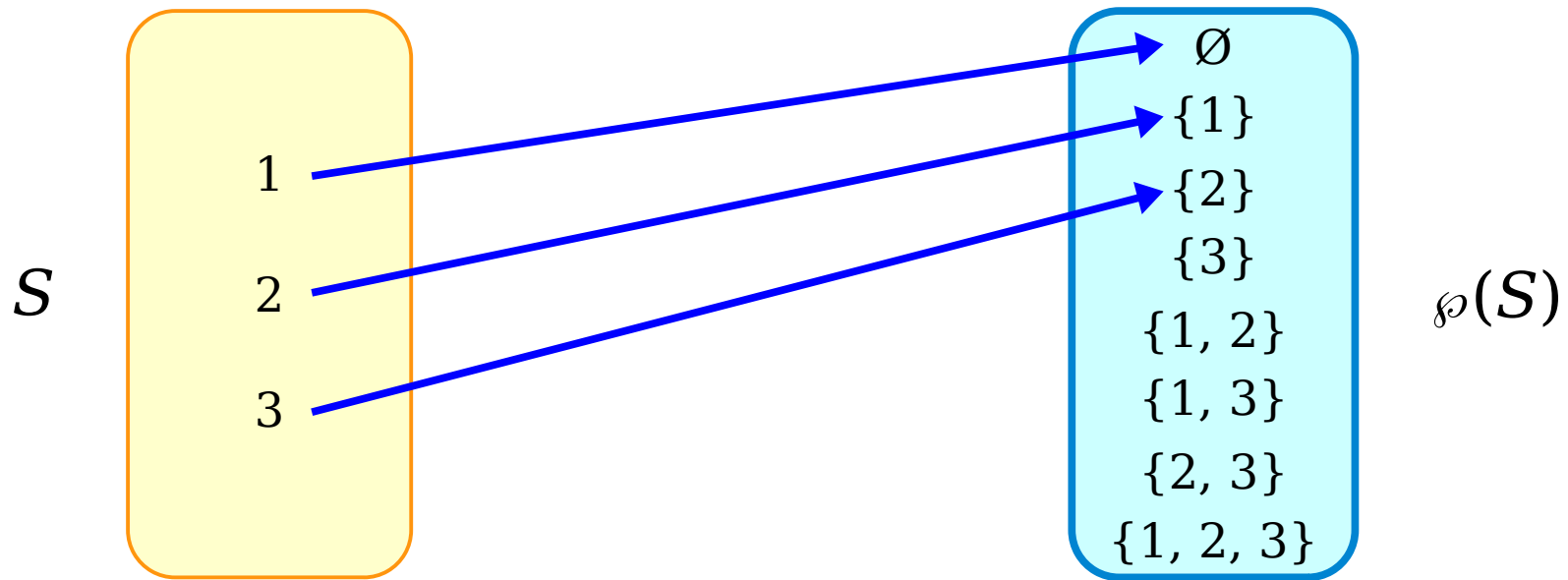
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



There are a lot of ways we could do this.



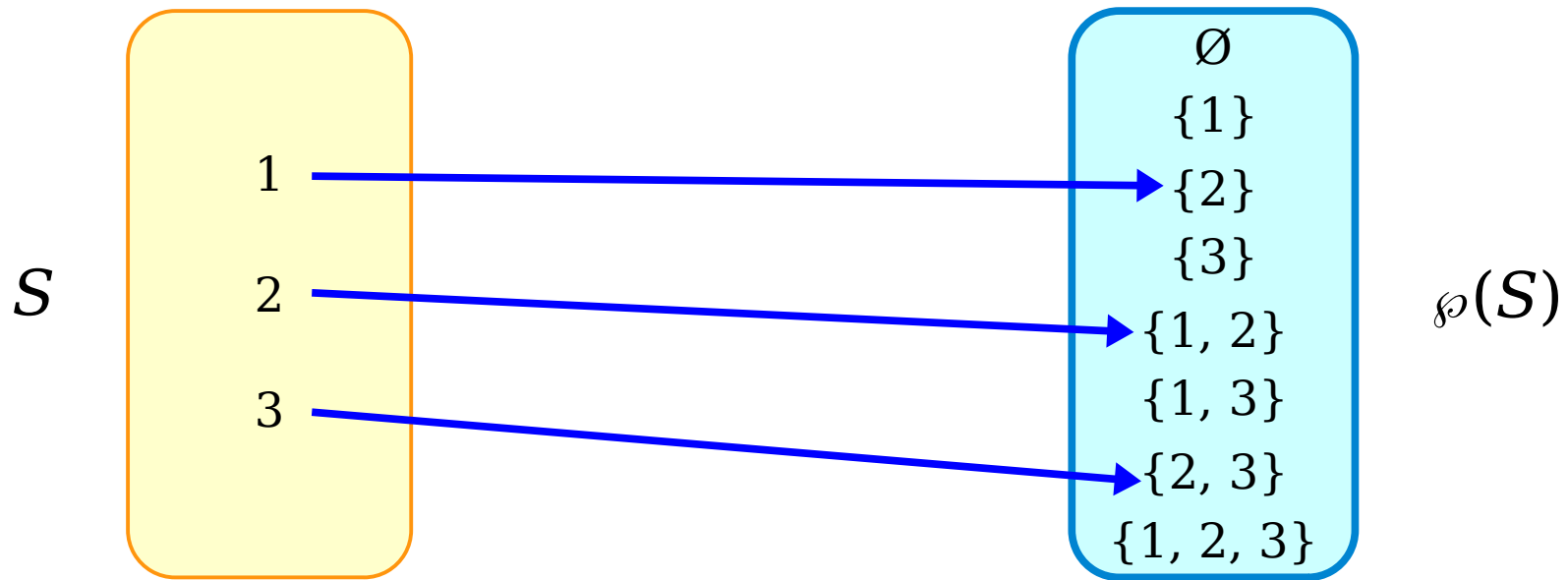
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Here's one possible injection.



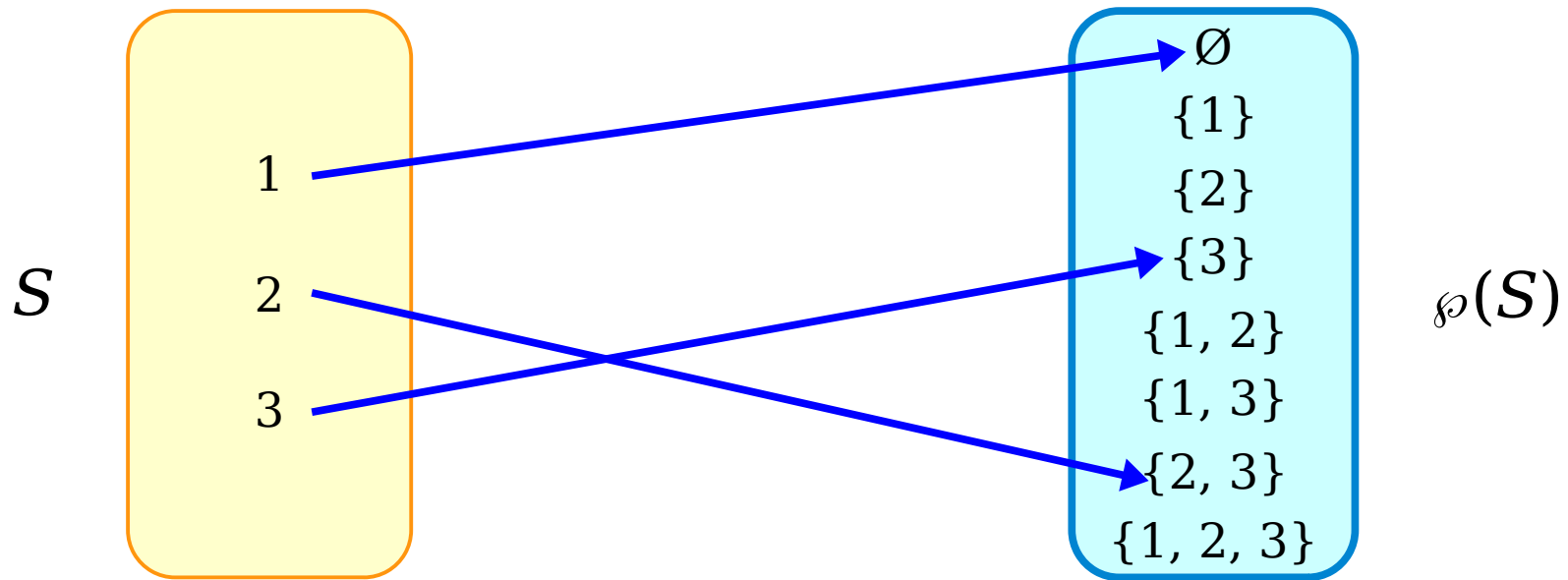
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Here's another!



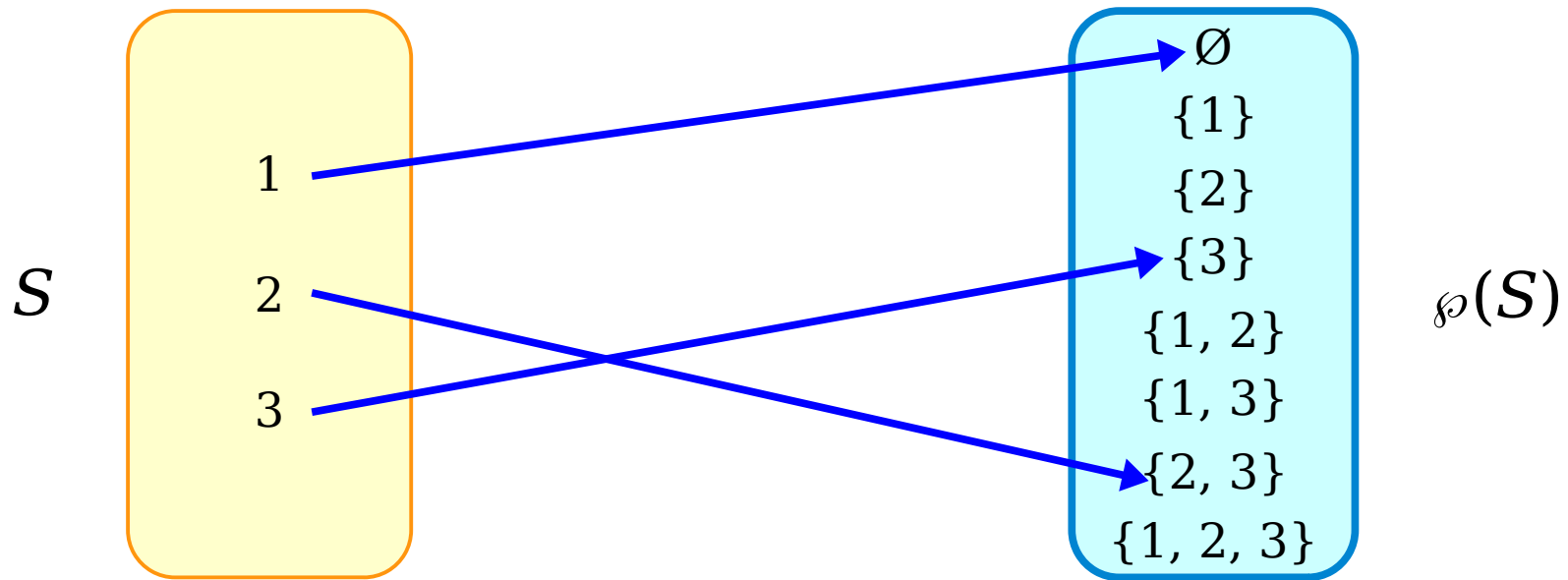
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



And here's one more.



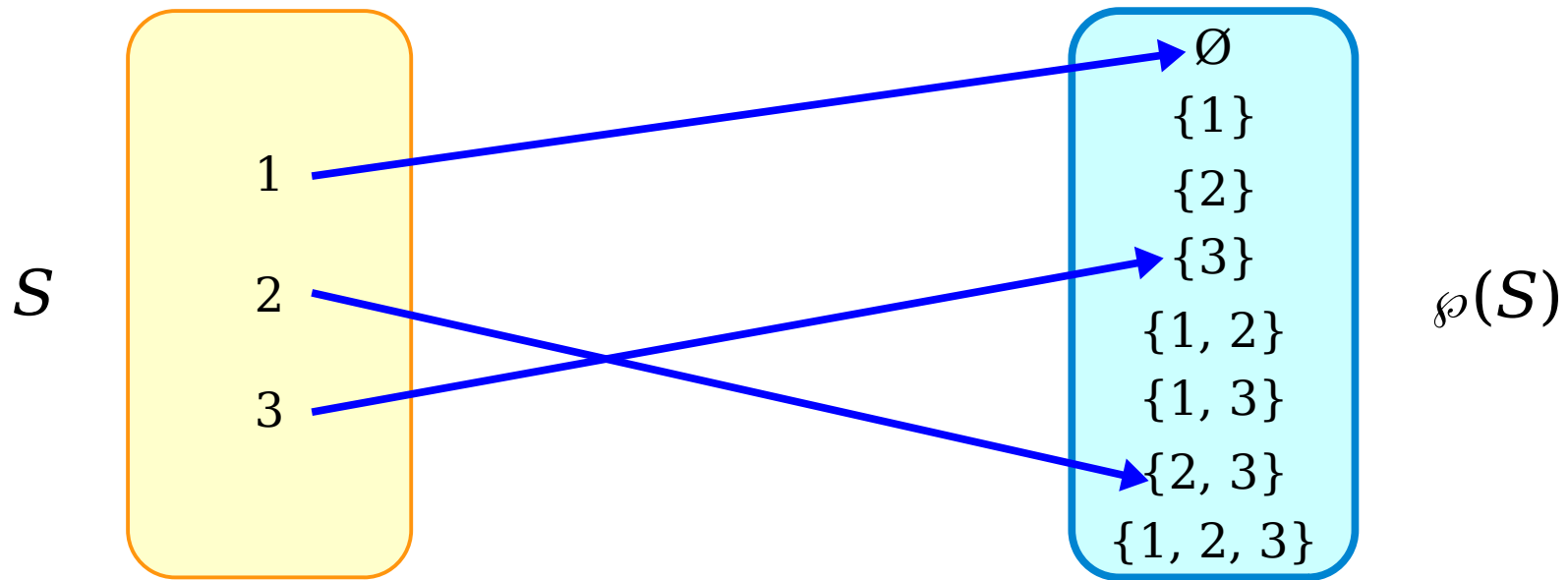
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



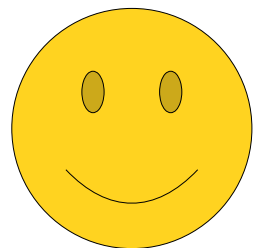
Now, ultimately, we're not trying to show that for this particular choice of S that an injection like this exists.



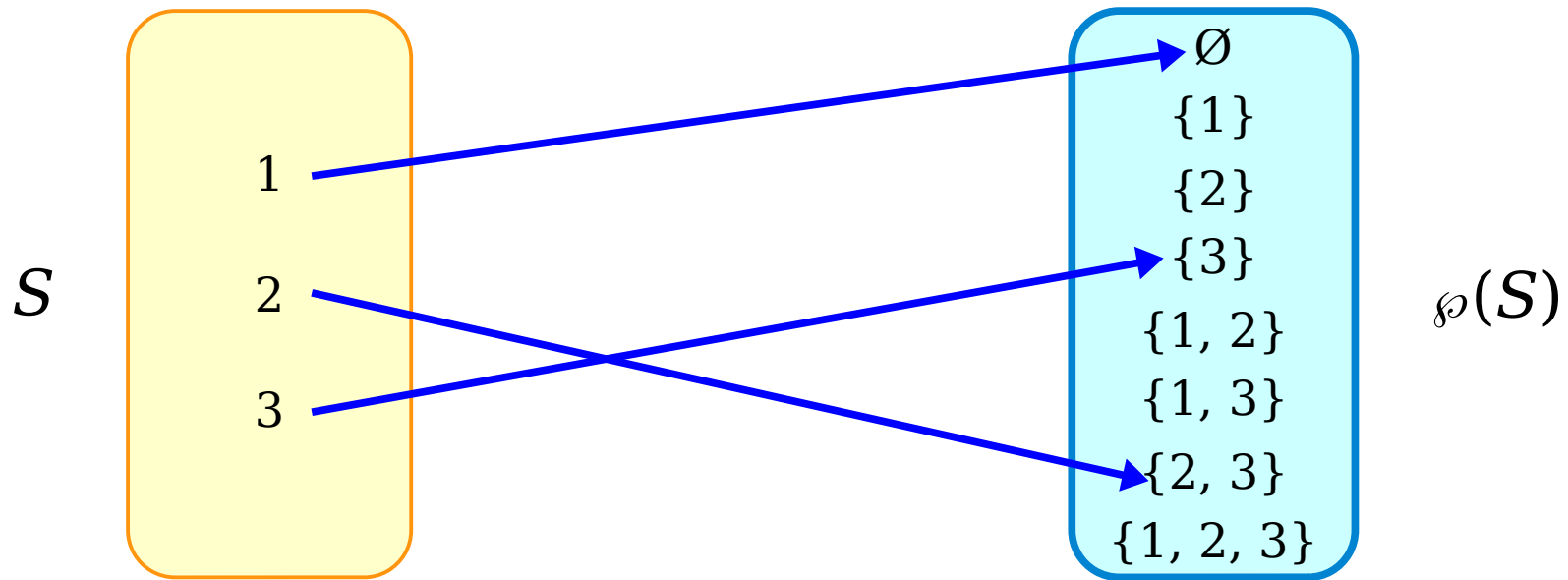
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



Rather, we're trying to show that no matter what set S we pick, we can always find an injection from S to $\wp(S)$.



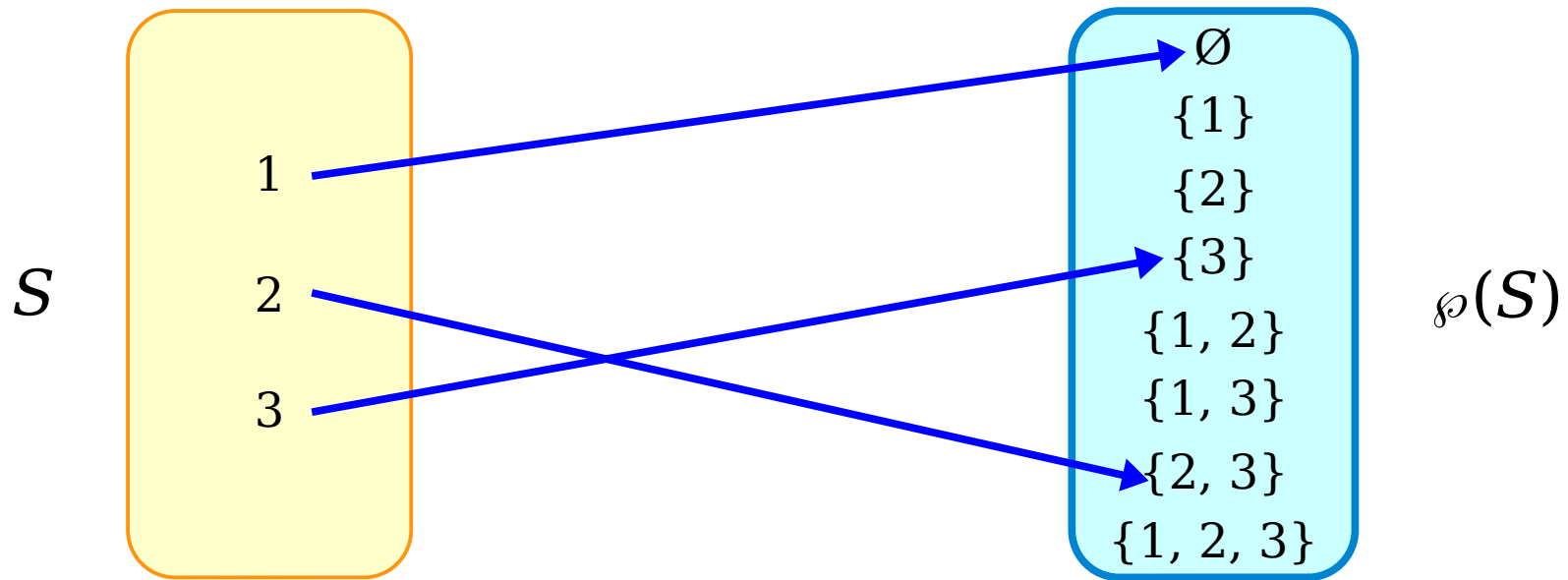
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



That means that we should try to look for some kind of pattern we can use that will always let us find an injection that works.



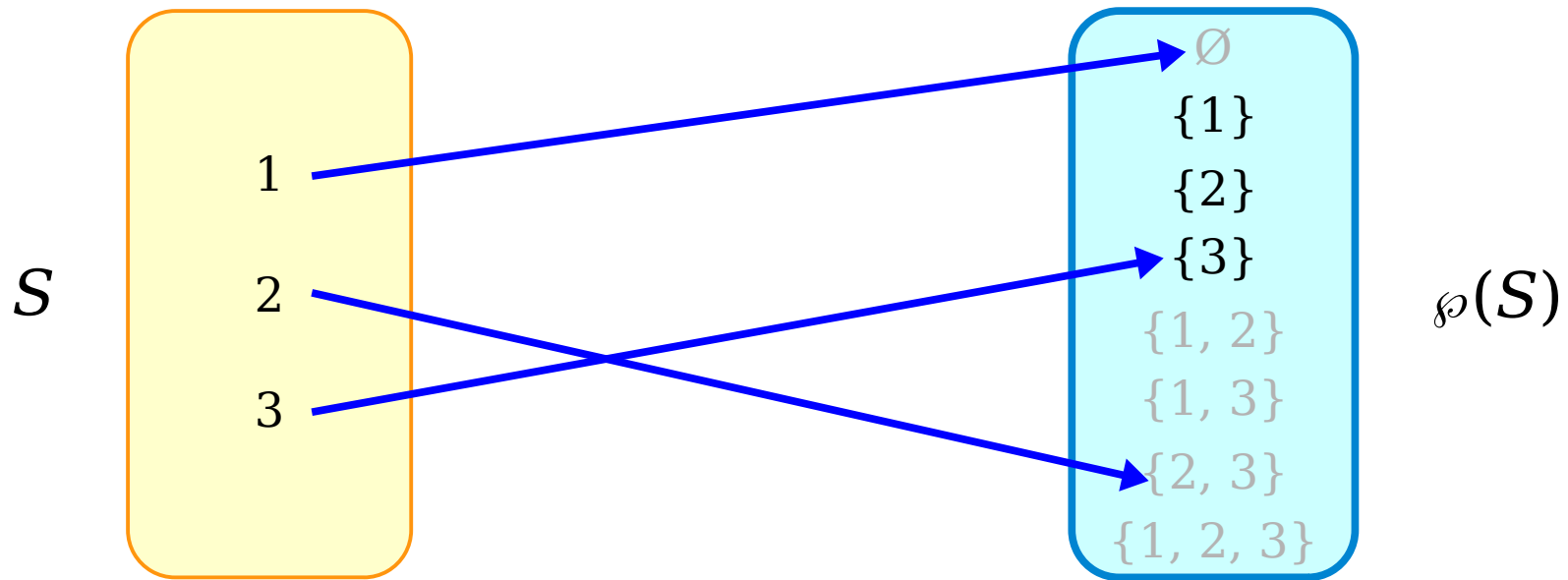
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



One observation that's useful for doing that here is that for every element of S ...



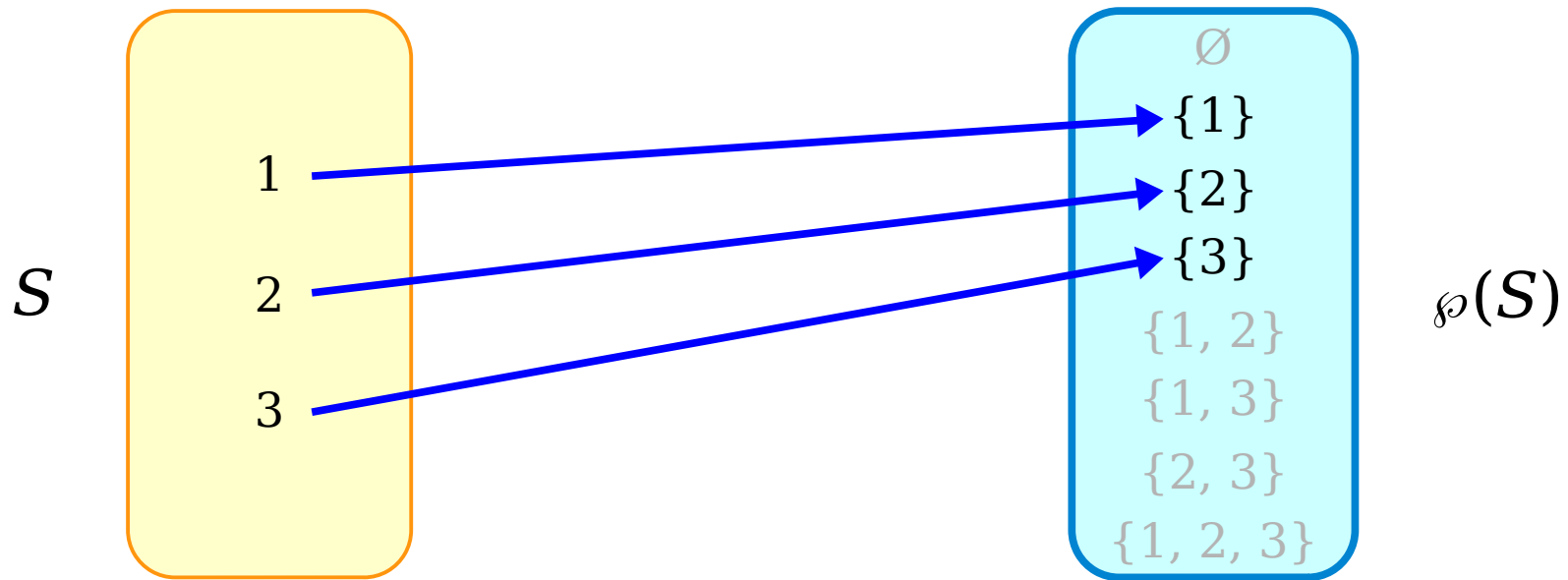
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



There's a singleton set containing just that element over in $\wp(S)$.



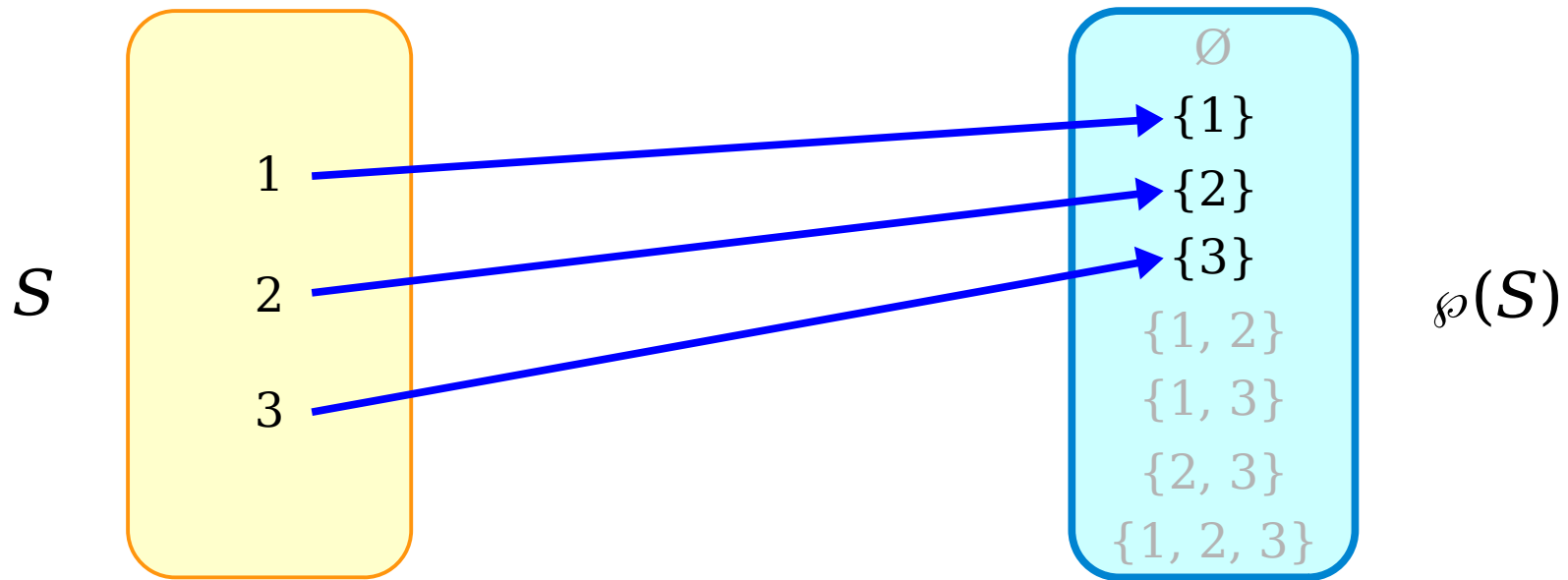
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



So perhaps we might want to look at
this particular injection.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

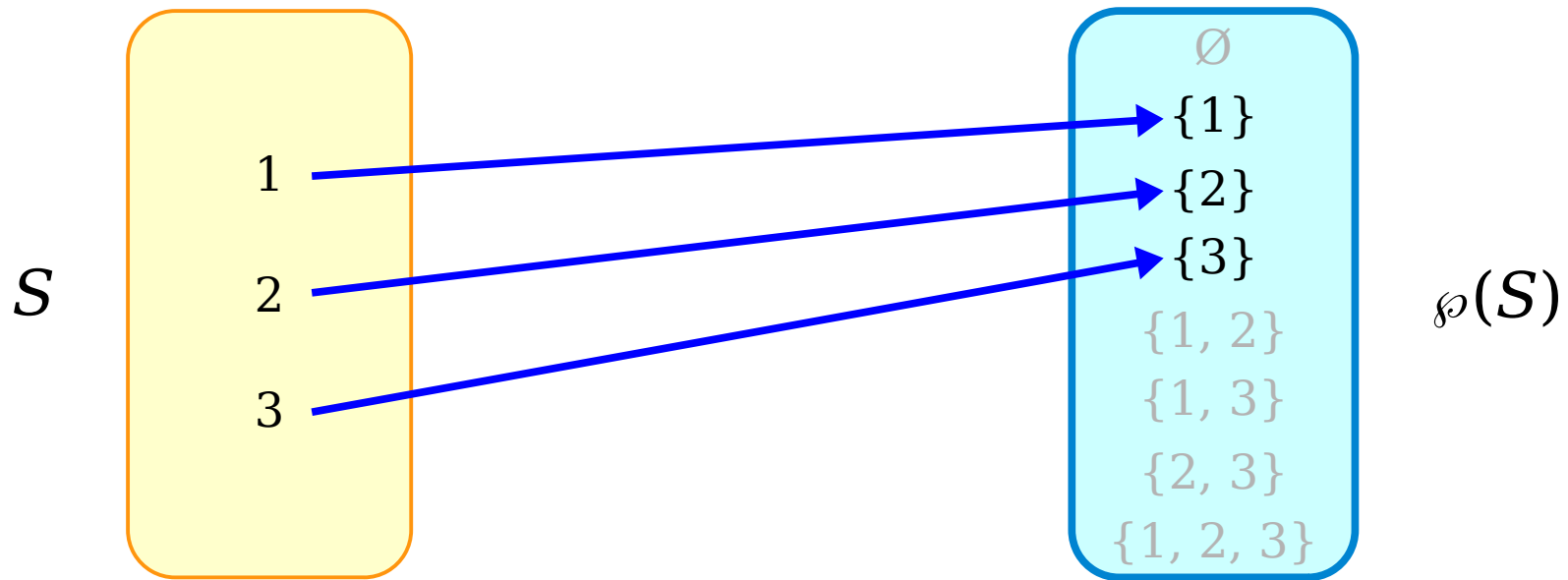


$$f : S \rightarrow \wp(S)$$
$$f(x) = \{x\}$$

This is a nice injection because there's a clean rule associating elements of S with elements of $\wp(S)$.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

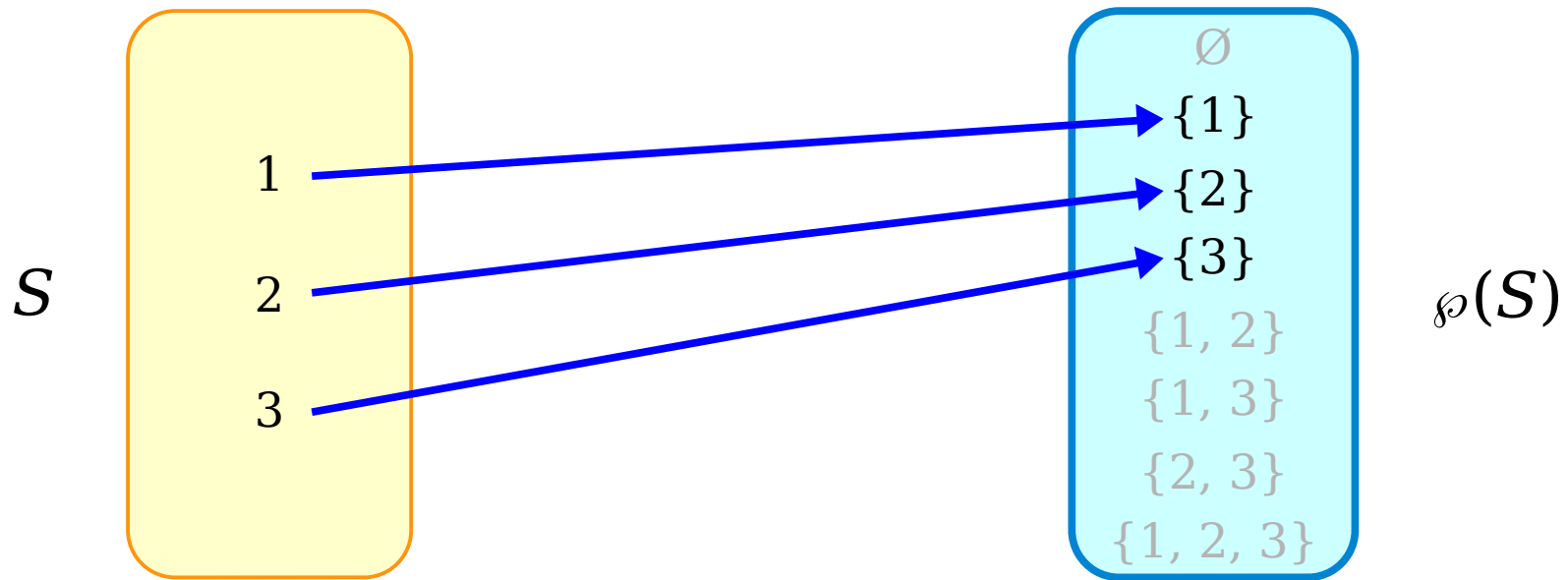


$$f : S \rightarrow \wp(S)$$
$$f(x) = \{x\}$$

With a little thought, we can see that no matter what set we choose for S , this function will always be well-defined.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f : S \rightarrow \wp(S)$$
$$f(x) = \{x\}$$

So if we want to prove the above lemma, perhaps we should try showing that this particular function always works as an injection from S to $\wp(S)$.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Proof: Let S be any set and consider the function $f : S \rightarrow \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.

Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required. ■

Here's one possible proof of this result. It follows the general pattern for proving that a function is injective, just using this particular choice of f .



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Proof: Let S be any set and consider the function $f : S \rightarrow \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.

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sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required. ■

I recommend taking a few minutes to read over this proof to make sure you understand how it works - it's a great exercise!



So, how did we get here, and where are we going?



If S is a set, then $|S| < |\wp(S)|$

This is the ultimate statement we're trying to prove.



There is an injection $f: S \rightarrow \wp(S)$
There is *no* bijection $f: S \rightarrow \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

As we saw earlier, this means that we need to prove these two statements.



There is an injection $f: S \rightarrow \wp(S)$ ✓
There is *no* bijection $f: S \rightarrow \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

We just finished that first step - great!



There is an injection $f : S \rightarrow \wp(S)$ ✓
There is *no* bijection $f : S \rightarrow \wp(S)$

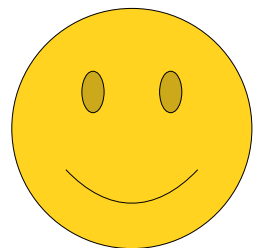
If S is a set, then $|S| < |\wp(S)|$

Now, we need to prove this part... and
that's where we're going to use a
diagonal argument.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

This is the statement that we now need to prove.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

The proof that we're going to do here is, essentially, a more rigorous version of the one we did on the first day of class.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

Let's start off by, briefly, reviewing
how that proof works.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0

x_1

x_2

x_3

x_4

x_5

...

Let's imagine that we have a set S
and that its elements are x_0, x_1, x_2 , etc.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0

x_1

x_2

x_3

x_4

x_5

...

Generally speaking, not all sets are going to look like this. Some might not have infinitely many elements. Others, like \mathbb{R} , have too many elements to assign a number to each one!



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0

x_1

x_2

x_3

x_4

x_5

...

But that's okay for now. This is just a visual intuition, and we'll address that when we try to make everything super formal and airtight.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0

x_1

x_2

x_3

x_4

x_5

...

The result we're going to prove involves showing that no choice of a function f from S to $\wp(S)$ is a bijection.



Lemma 2: If $f : S \rightarrow {}^{\mathcal{P}}(S)$ is a function, then it is not bijective.

x_0

x_1

x_2

x_3

x_4

x_5

...

Just to explore things a bit, let's choose some random function f and see if we notice anything about it.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

Here's one example function f . There's no particular pattern here in the function we picked – it's just a random function to help us build an intuition.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

Our ultimate goal is to prove that this
function is not bijective.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

That's going to be a challenge, since we can't assume that this function follows any nice pattern.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

Our approach will be to show that this function is not surjective. If we can do that, then we can say with certainty that f is not bijective.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

To do so, we're going to have to find a way to show that there's some subset of S that isn't covered by our function.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

Visually, that means that we need to find a set that's not paired with anything in the left-hand column.



Lemma 2: If $f: S \rightarrow \wp(S)$ is a function, then it is not bijective.

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

...

To do so, we're going to use a trick
invented by Georg Cantor, which will
require us to redraw this picture a bit.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$$x_0 \longrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longrightarrow \{ x_4, \dots \}$$

$$x_3 \longrightarrow \{ x_1, x_3, x_4, \dots \}$$

$$x_4 \longrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

\dots

We're going to start off by writing the elements of S across the top, like this.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

\dots

We can then imagine spacing out the elements that are in each of these sets so that we start to have something that looks more like a grid.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

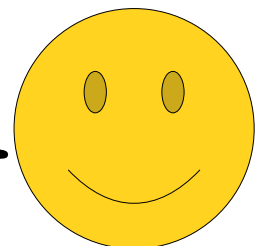
$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

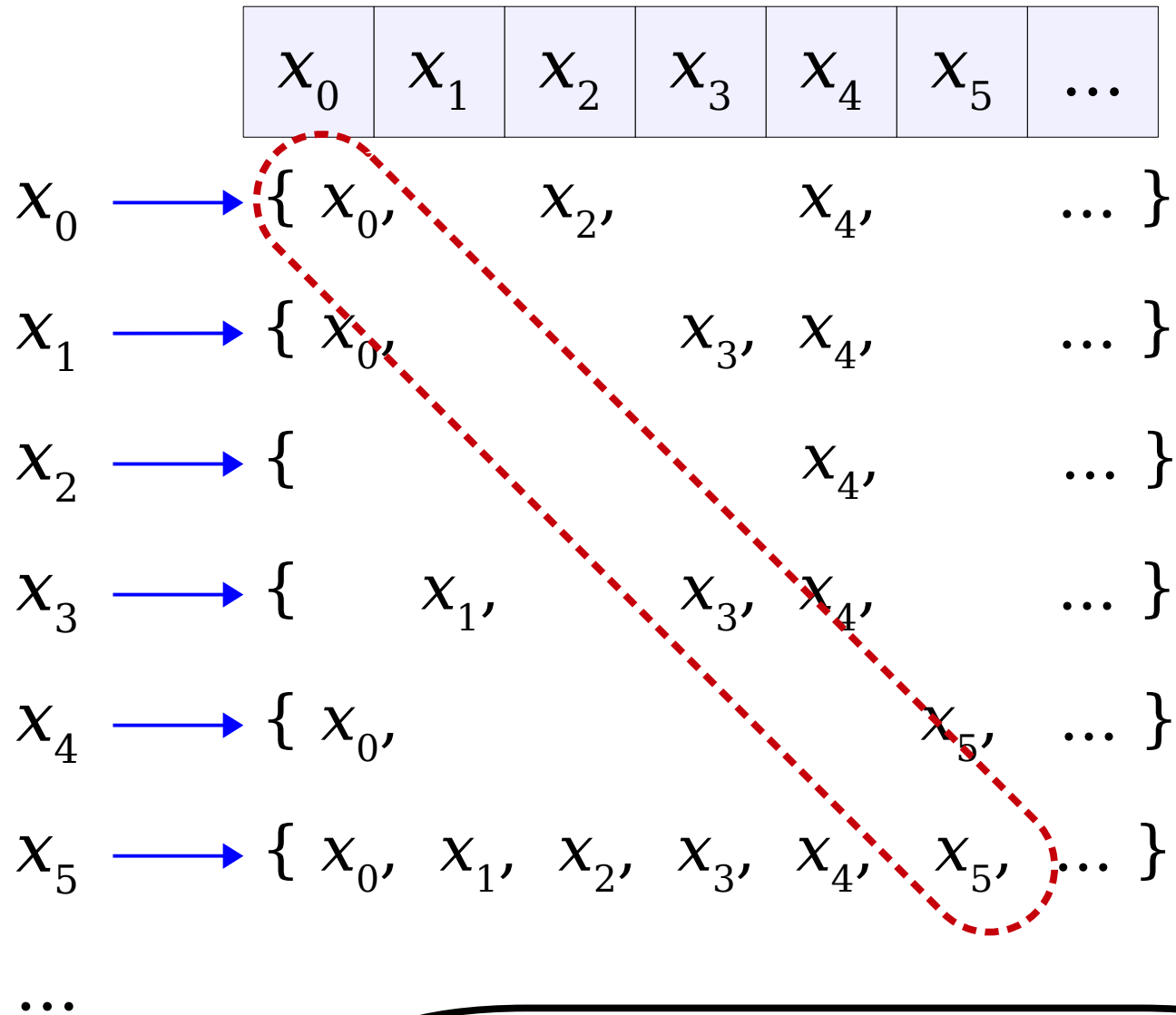
$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

...

Cantor's insight - which is the real trick behind the proof - is the following.



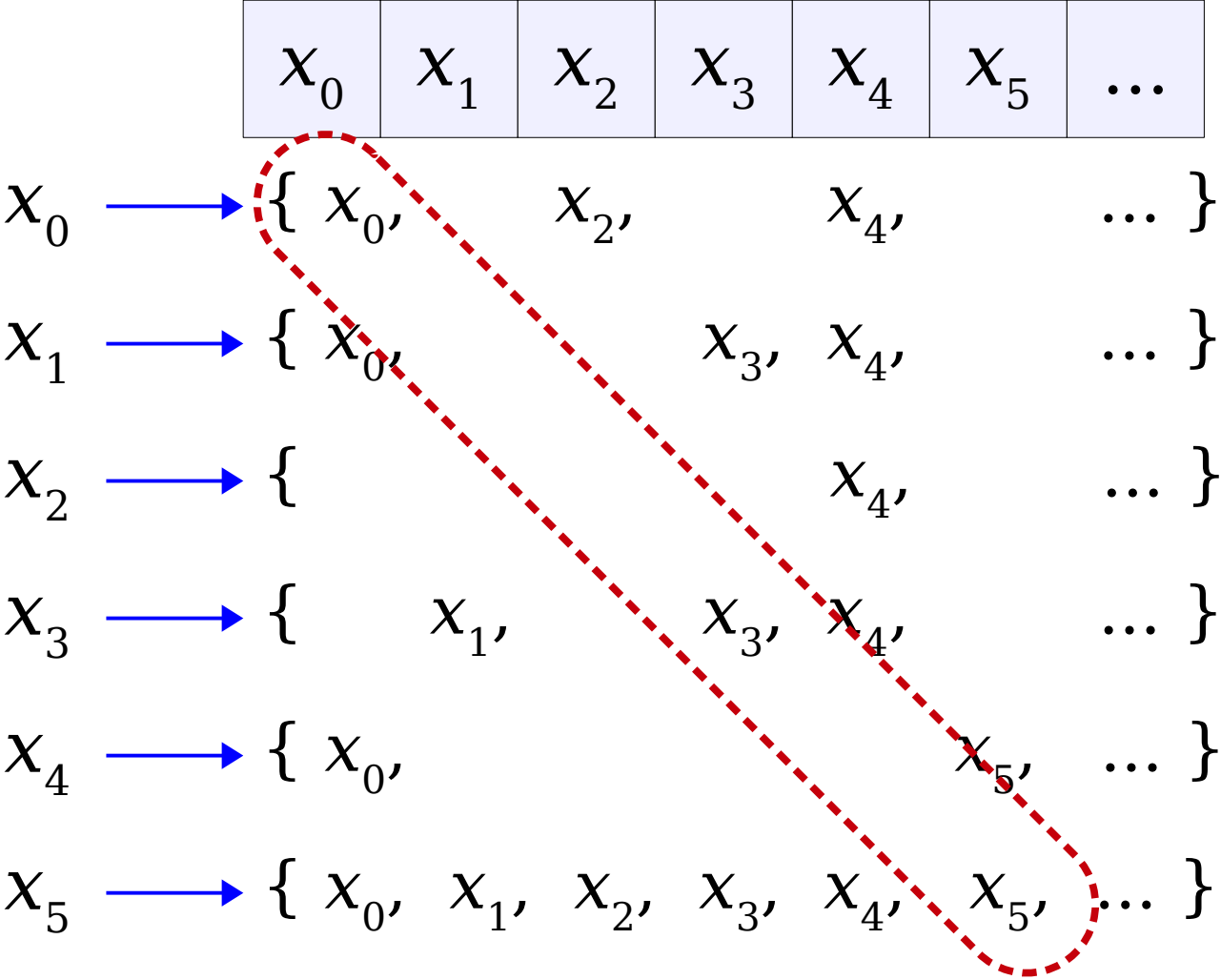
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



The trick, as you saw in lecture one, is to look at this main diagonal and use it to build a set that's not mapped to.



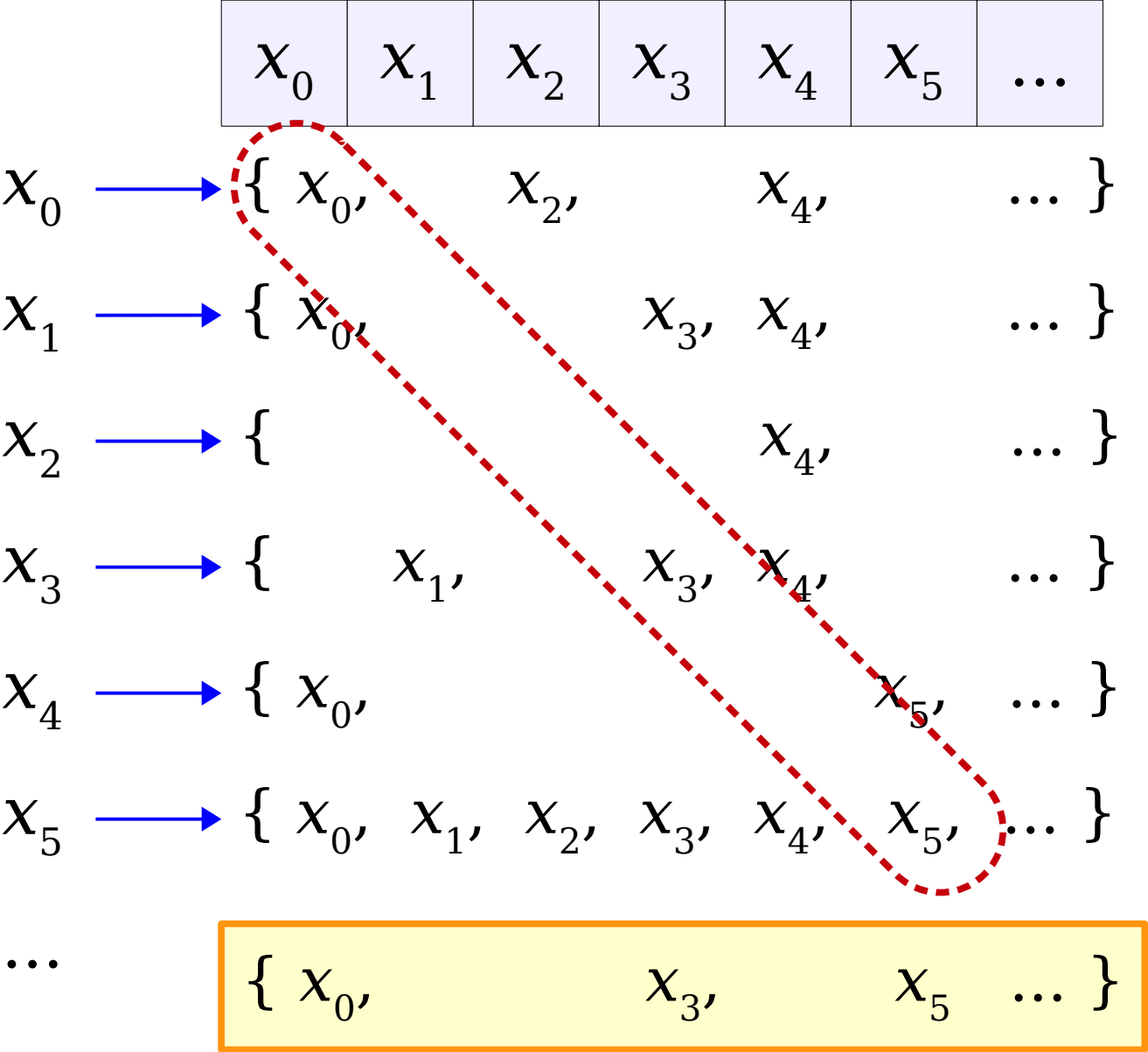
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



This is a two-step process.



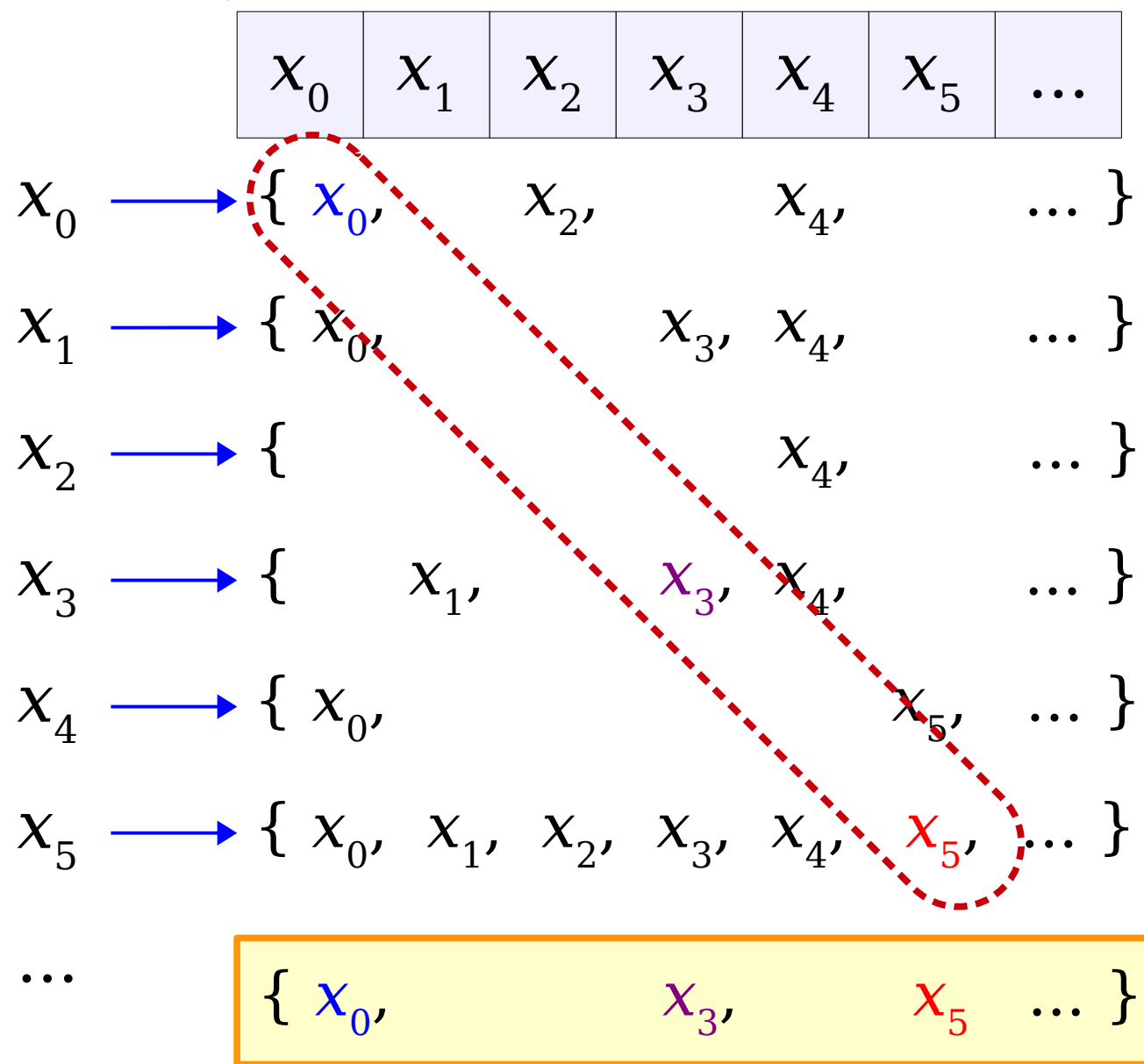
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



First, we're going to form a set of all the elements on the diagonal.



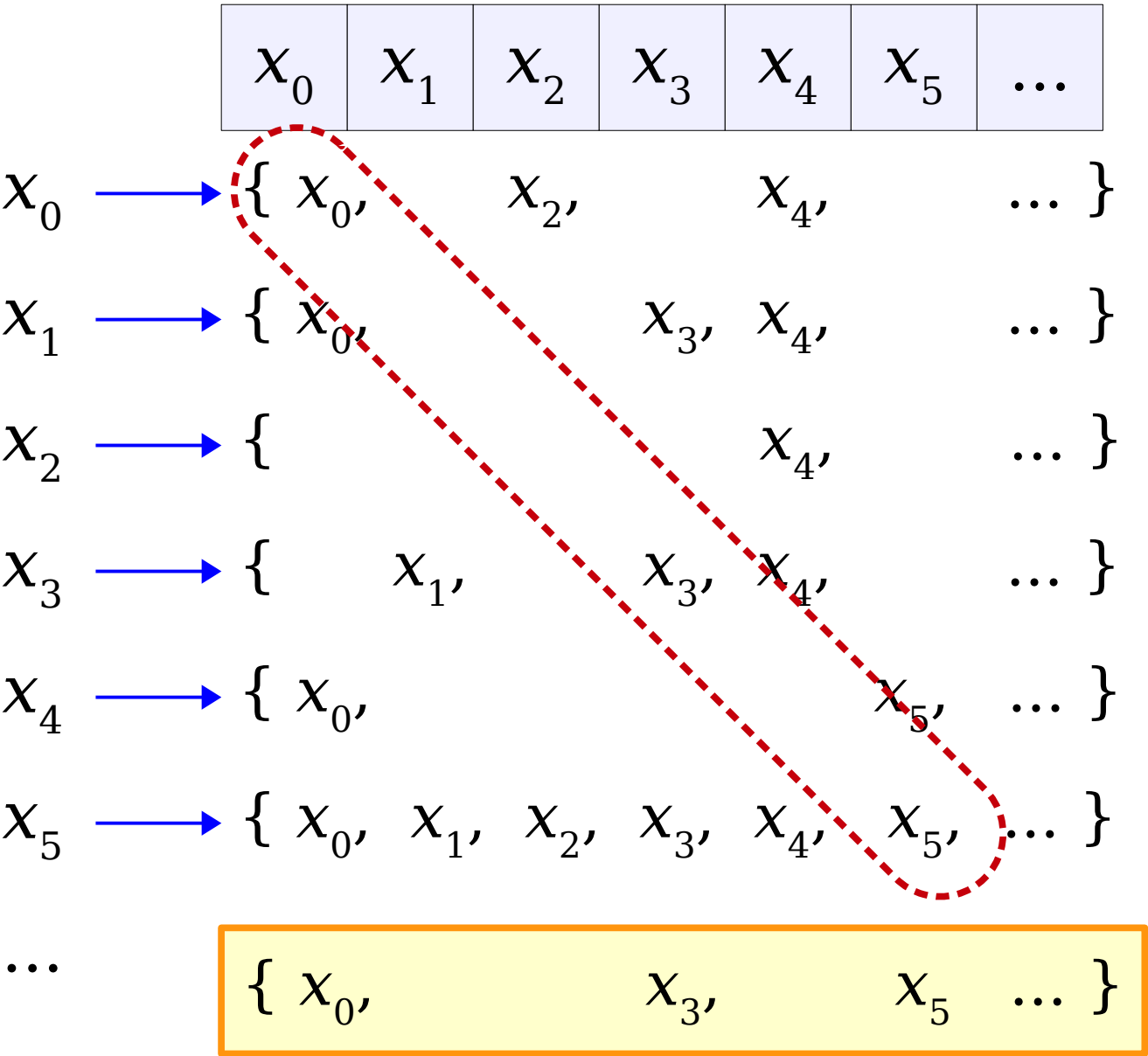
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



This color-coding hopefully makes it a bit easier to see where everything comes from.



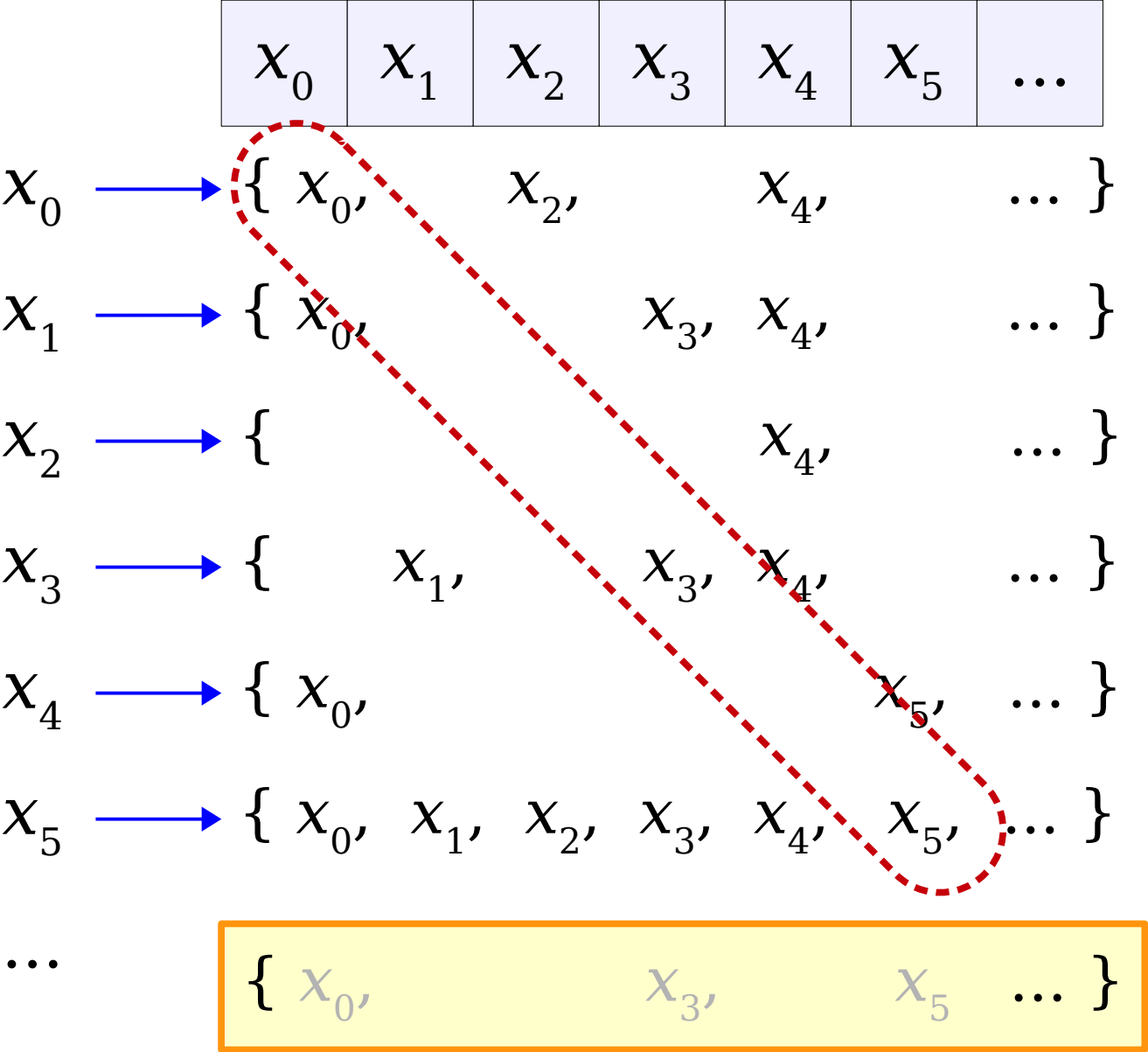
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



Next, we're going to "flip" this set.



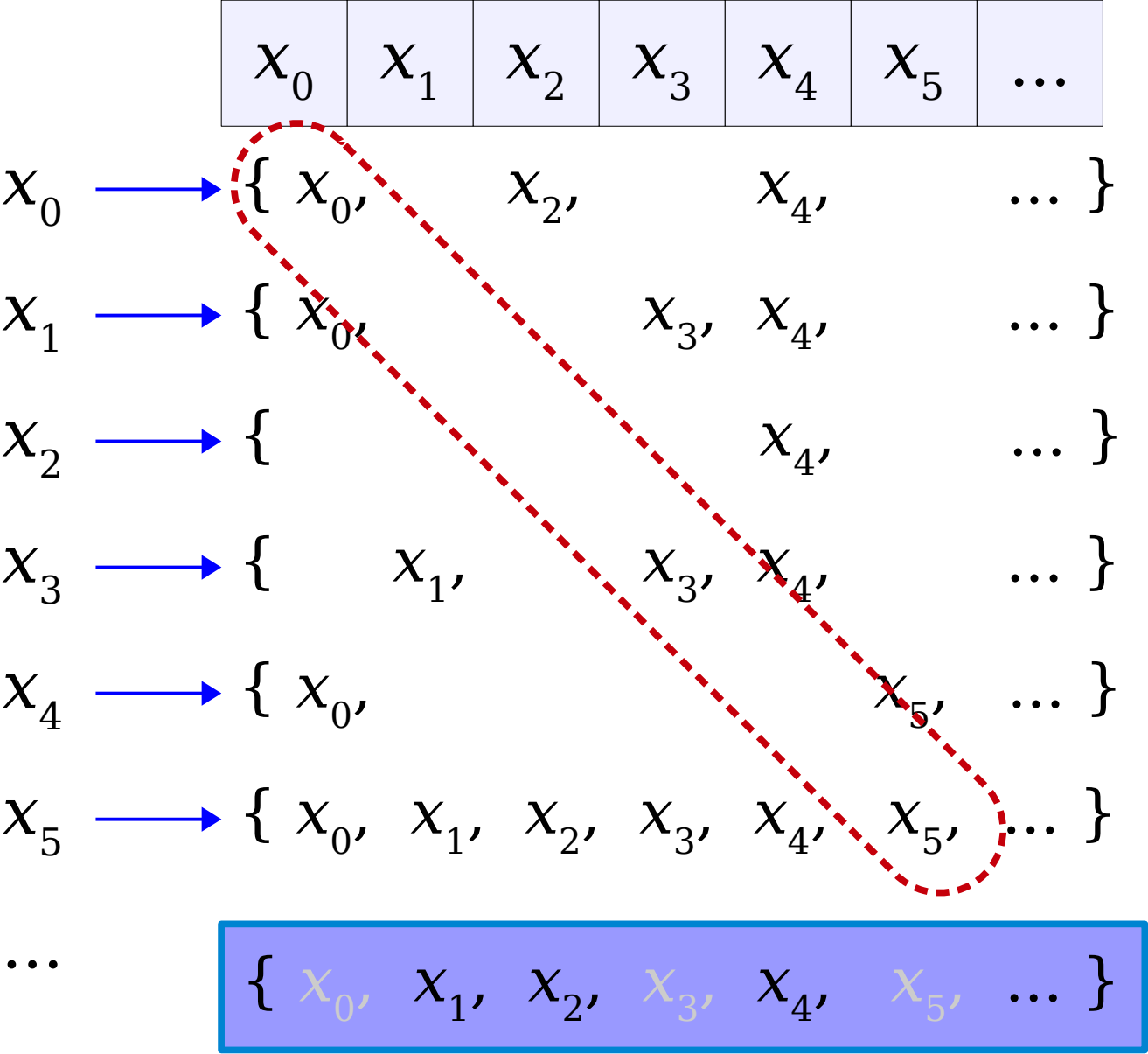
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



We're going to remove everything that
used to be in this set...



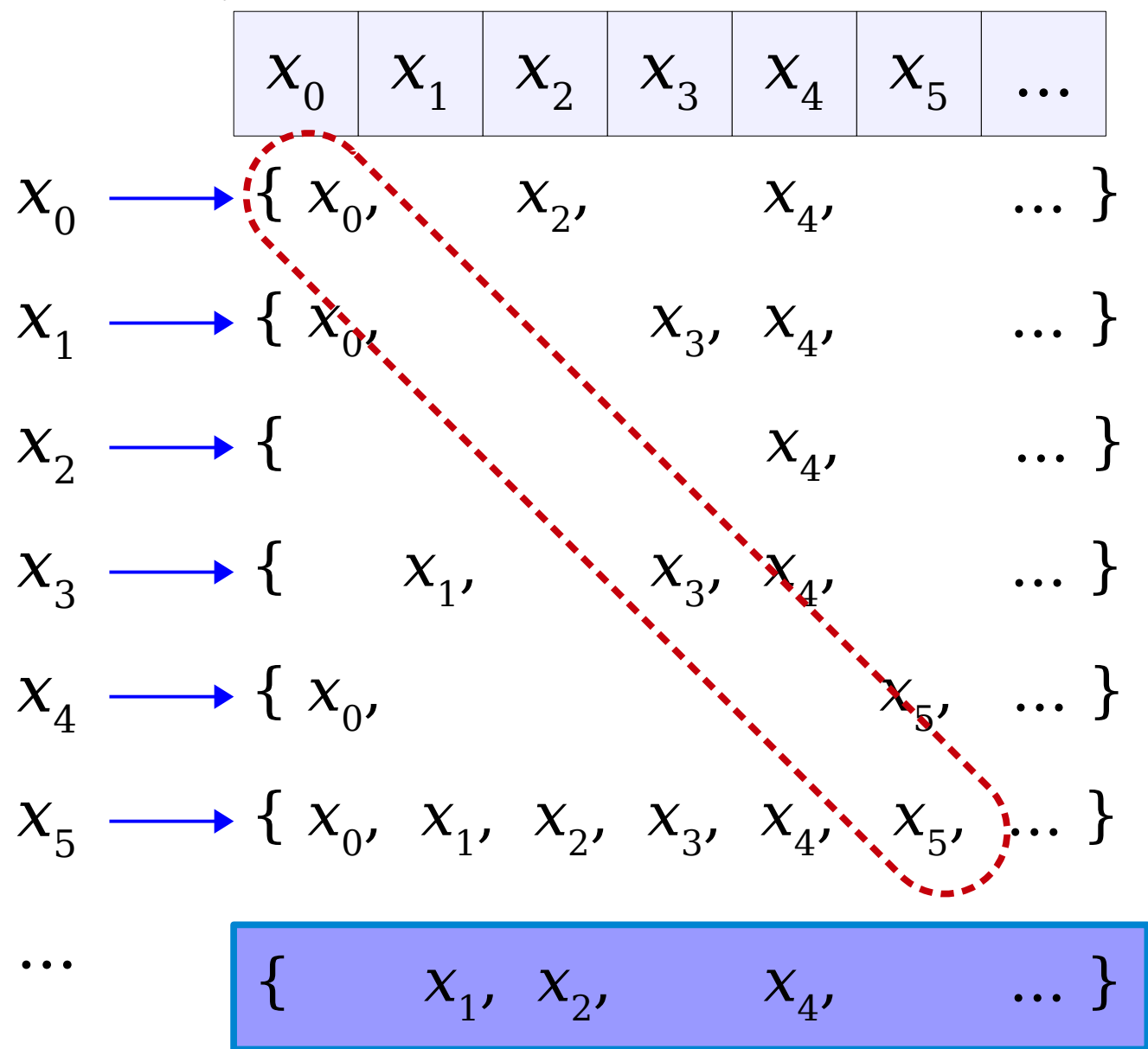
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



Then, we're going to add everything that was missing from the original set.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



This new set – which is called the diagonal set – is specifically built to be different from every row above.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

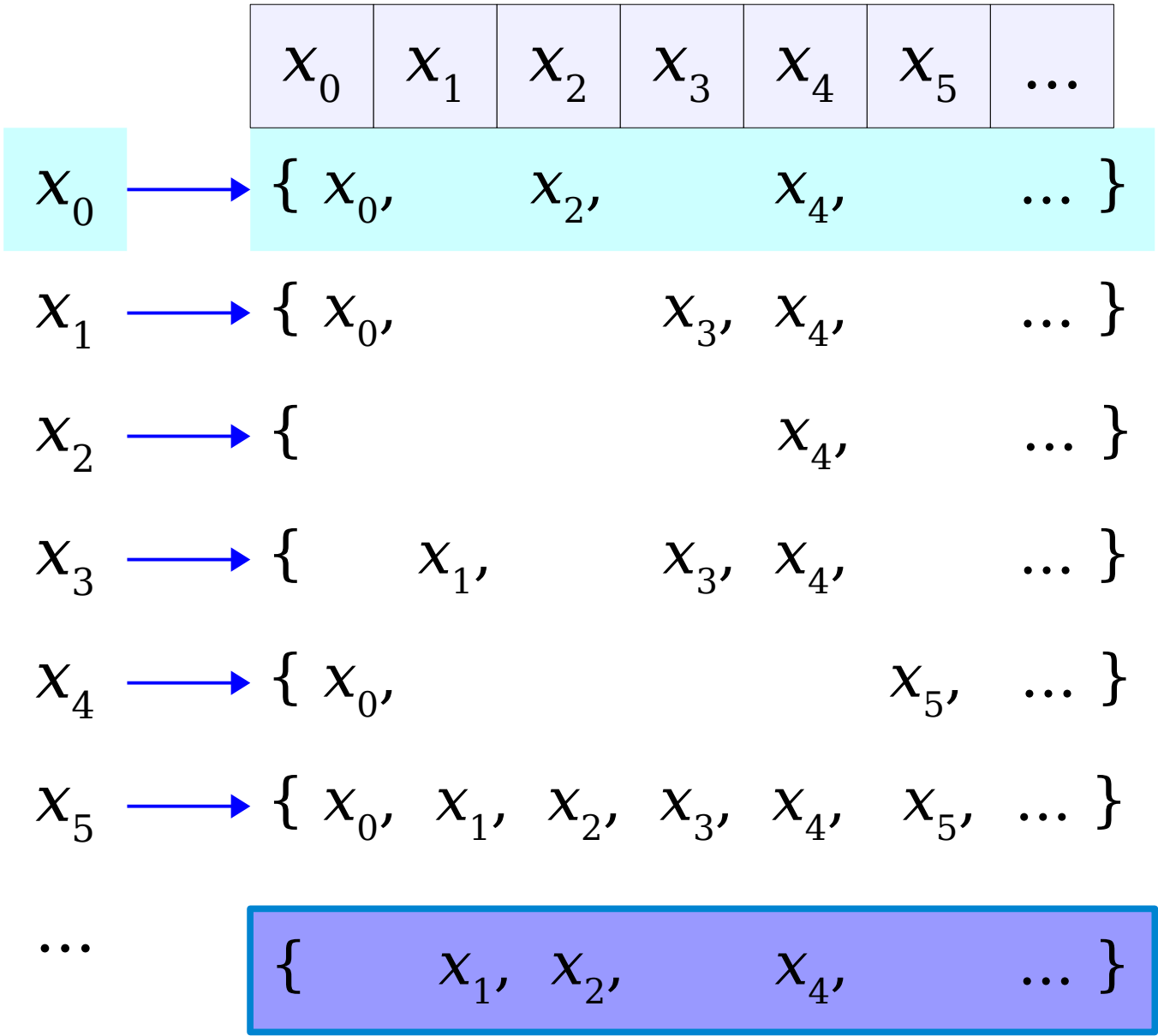
$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

\dots $\{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Let's see why this is.



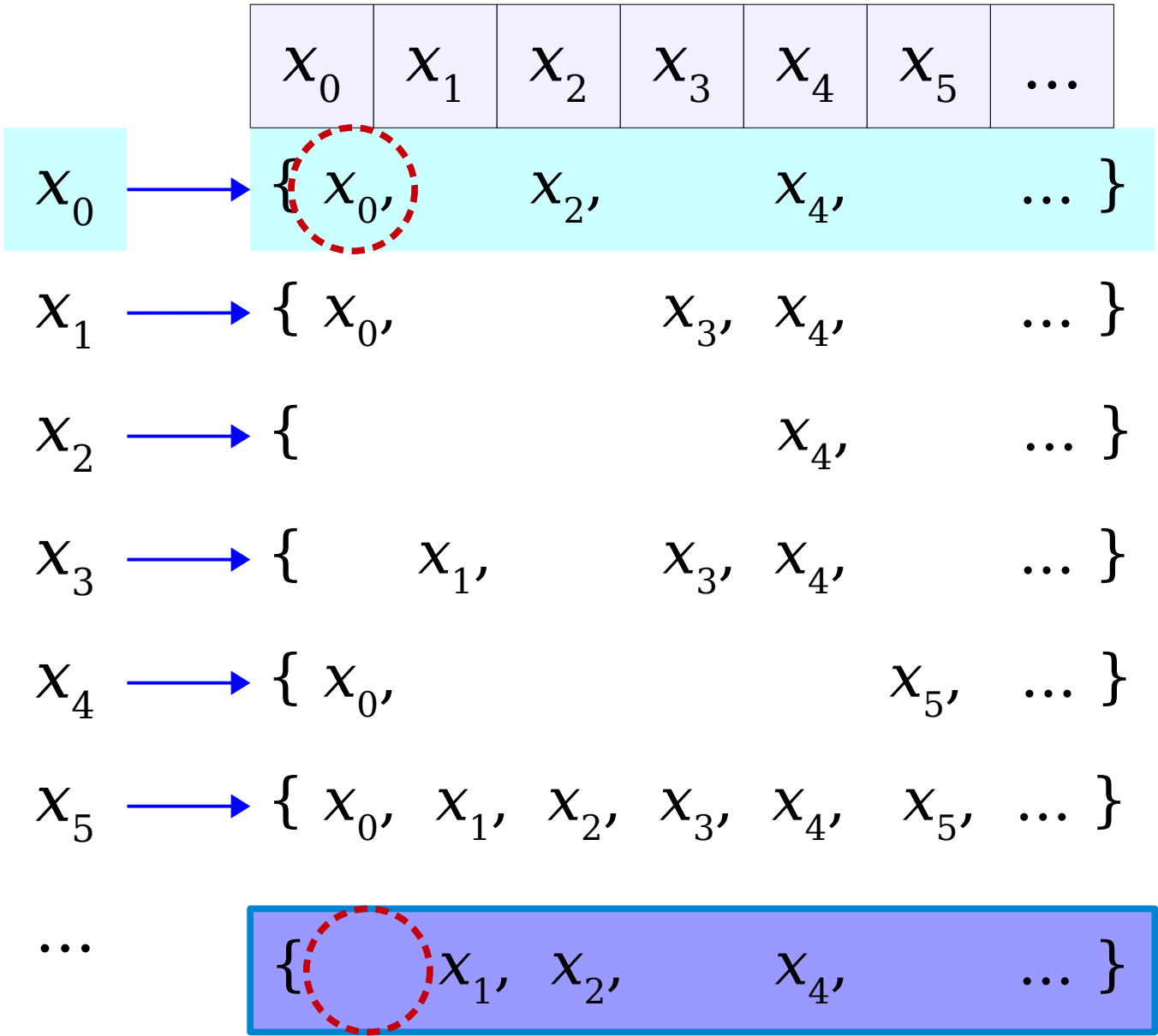
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



For example, we can ask whether this set is equal to the set matched with x_0 .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



But we can see that's not the case. The set paired with x_0 contains x_0 , but our diagonal set doesn't contain x_0 .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

So maybe this set is paired with x_1 ?



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \text{ } \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Well, that can't be the case, since the set paired with x_1 doesn't contain x_1 , but the diagonal set does contain x_1 .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad \bigcirc \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

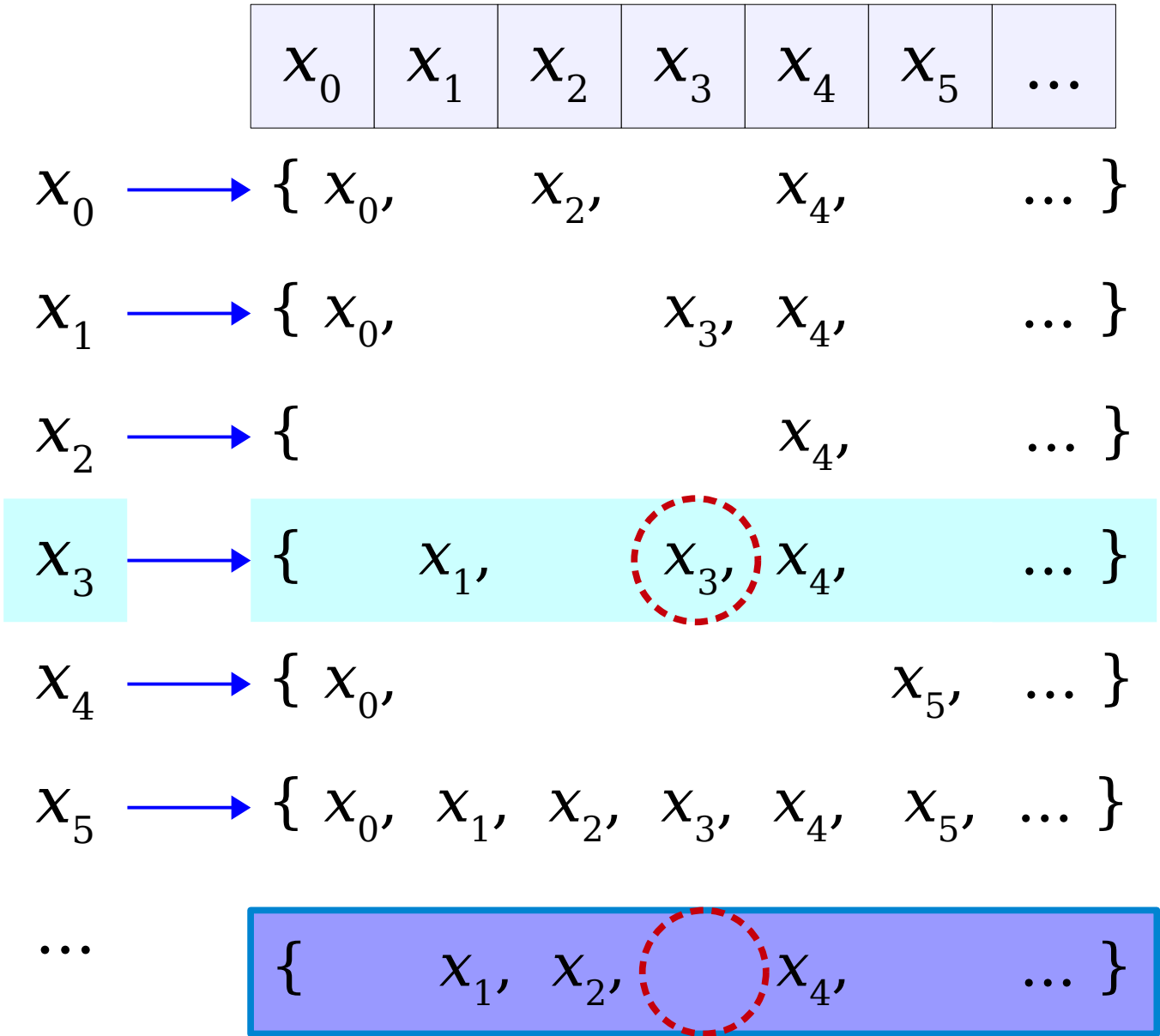
$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

... $\{ \quad x_1, \quad \bigcirc \quad x_4, \quad \dots \}$

More generally...



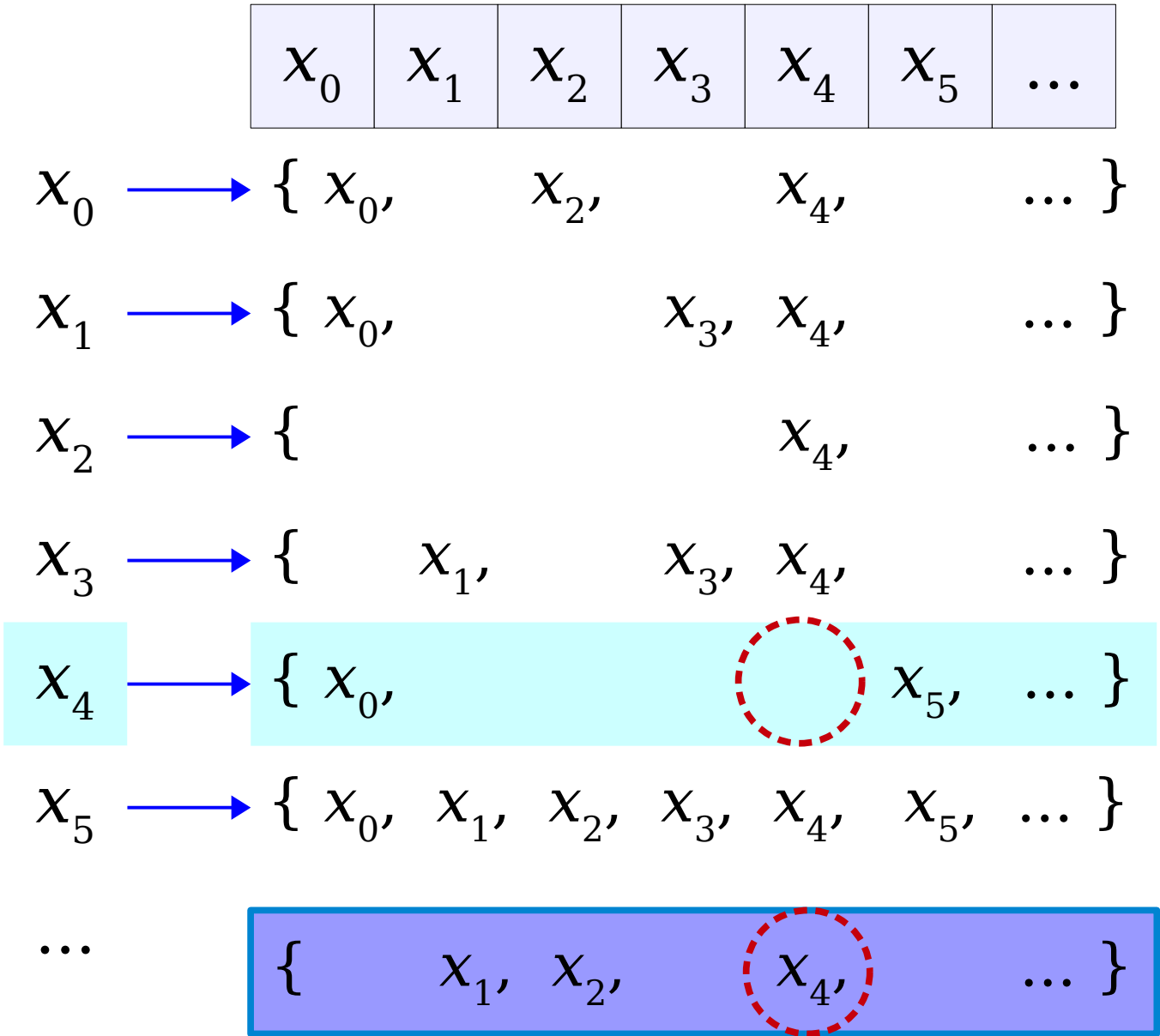
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



... as we move down the diagonal ...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



... we always see that there's a mismatch between the set in that row ...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$$x_0 \longrightarrow \{ x_0, \qquad x_2, \qquad x_4, \qquad \dots \}$$

$$x_1 \longrightarrow \{ x_0, \qquad \qquad x_3, x_4, \qquad \dots \}$$

$$x_2 \longrightarrow \{ \qquad \qquad \qquad x_4, \qquad \dots \}$$

$$x_3 \longrightarrow \{ \qquad x_1, \qquad x_3, x_4, \qquad \dots \}$$

$$x_4 \longrightarrow \{ x_0, \qquad \qquad \qquad x_5, \dots \}$$

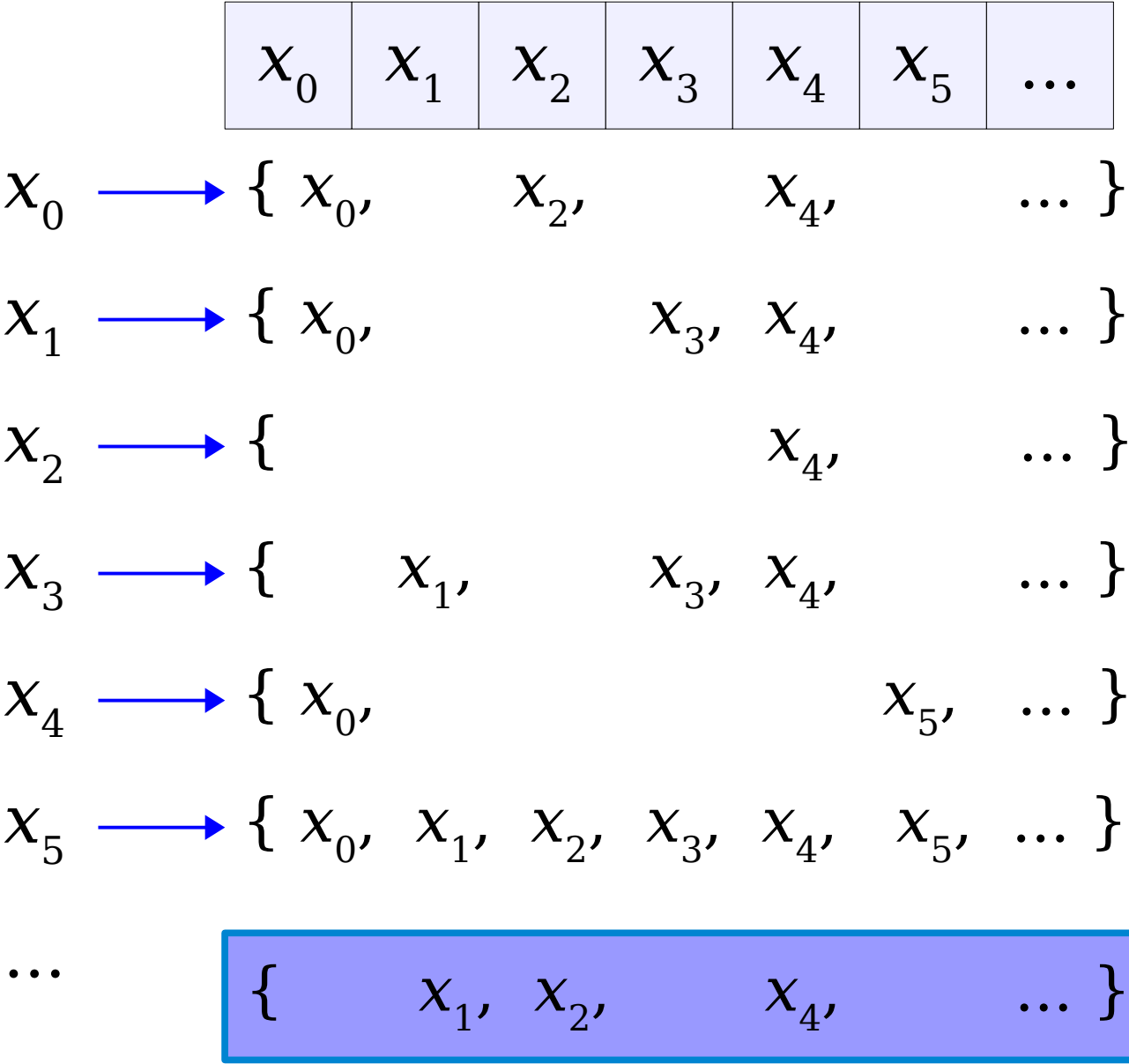
$$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$$\dots \qquad \{ \qquad x_1, x_2, \qquad x_4, \qquad \dots \}$$

... and the diagonal set we built.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



As a result, we know that this new set isn't paired with anything.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Formally speaking, if we call our set D ...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

... $\longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

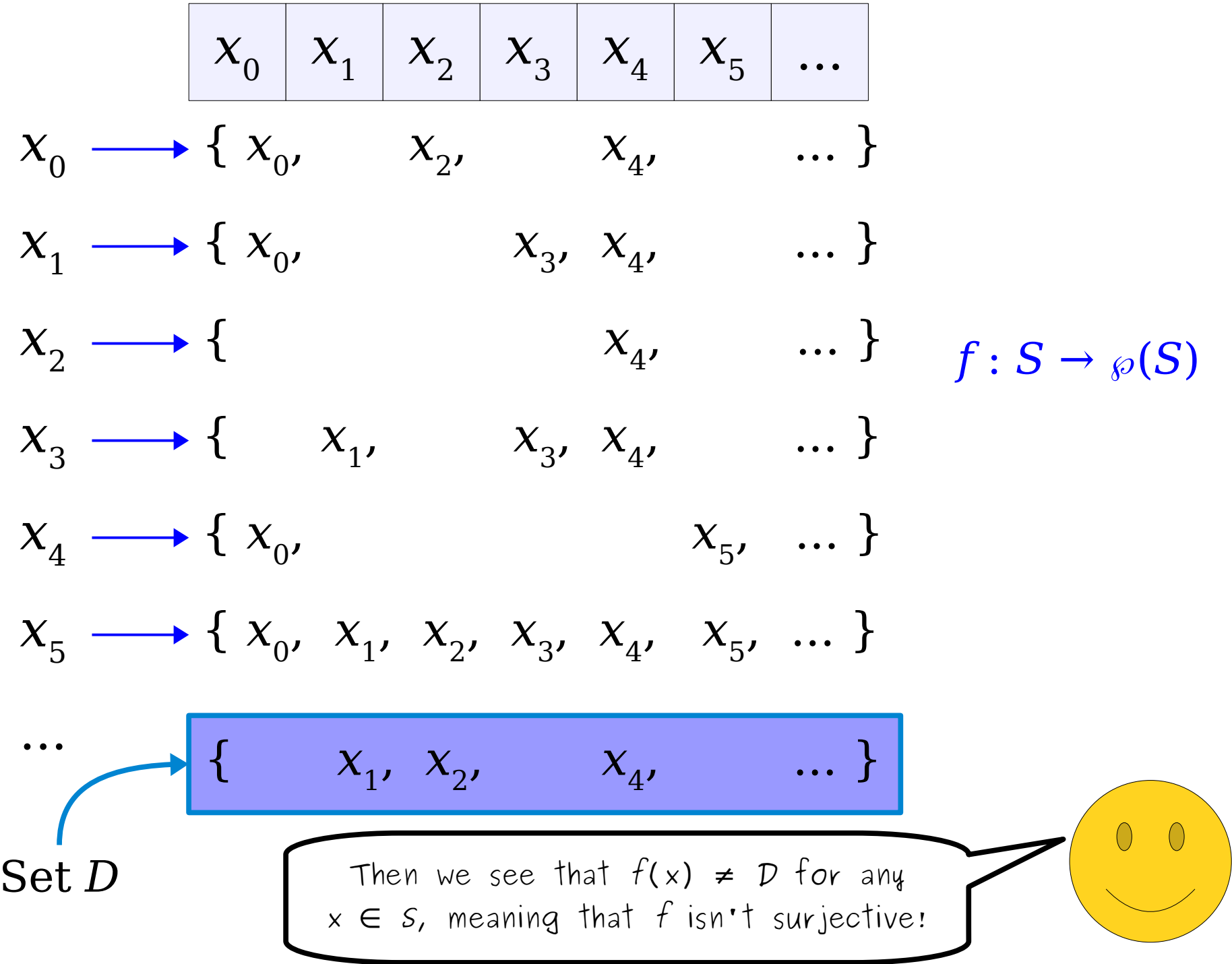
$f : S \rightarrow \wp(S)$

Set D

... and this particular function is f ...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$$

$$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$$

$$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$$

$$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$$

$$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$$

$$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$$

$$f : S \rightarrow \wp(S)$$

...

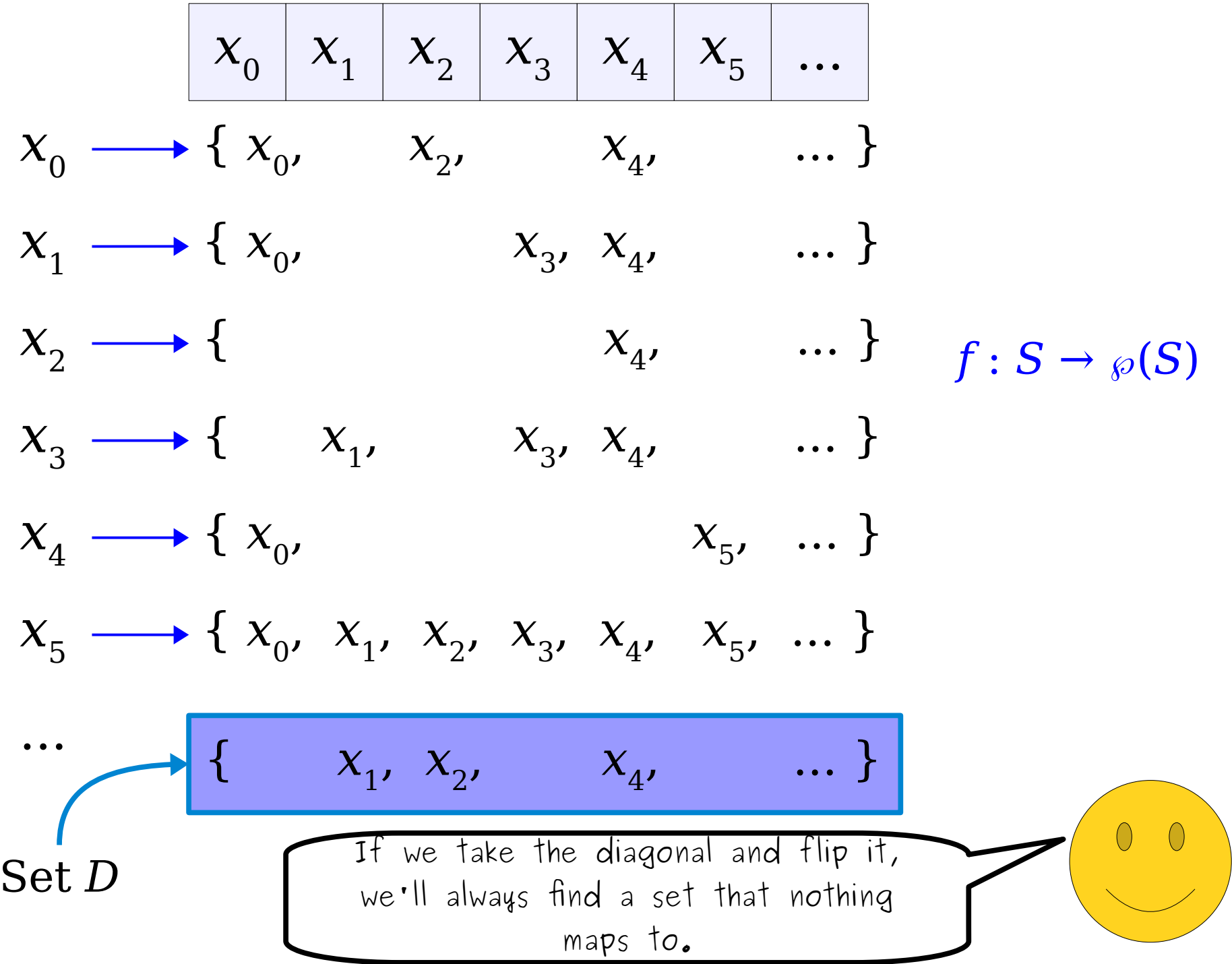
$\{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

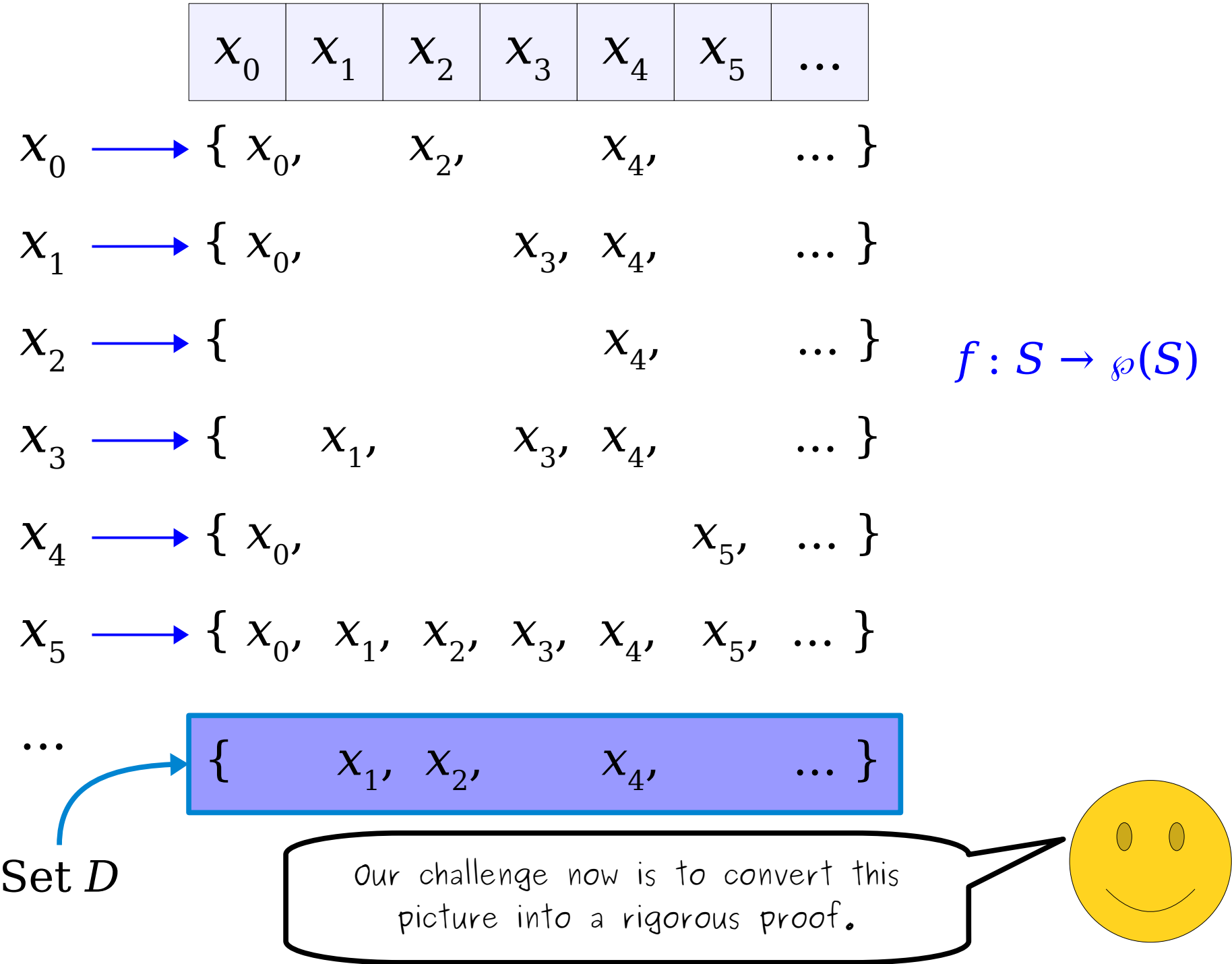
The beauty behind this construction is that it doesn't matter what choice of f we make.



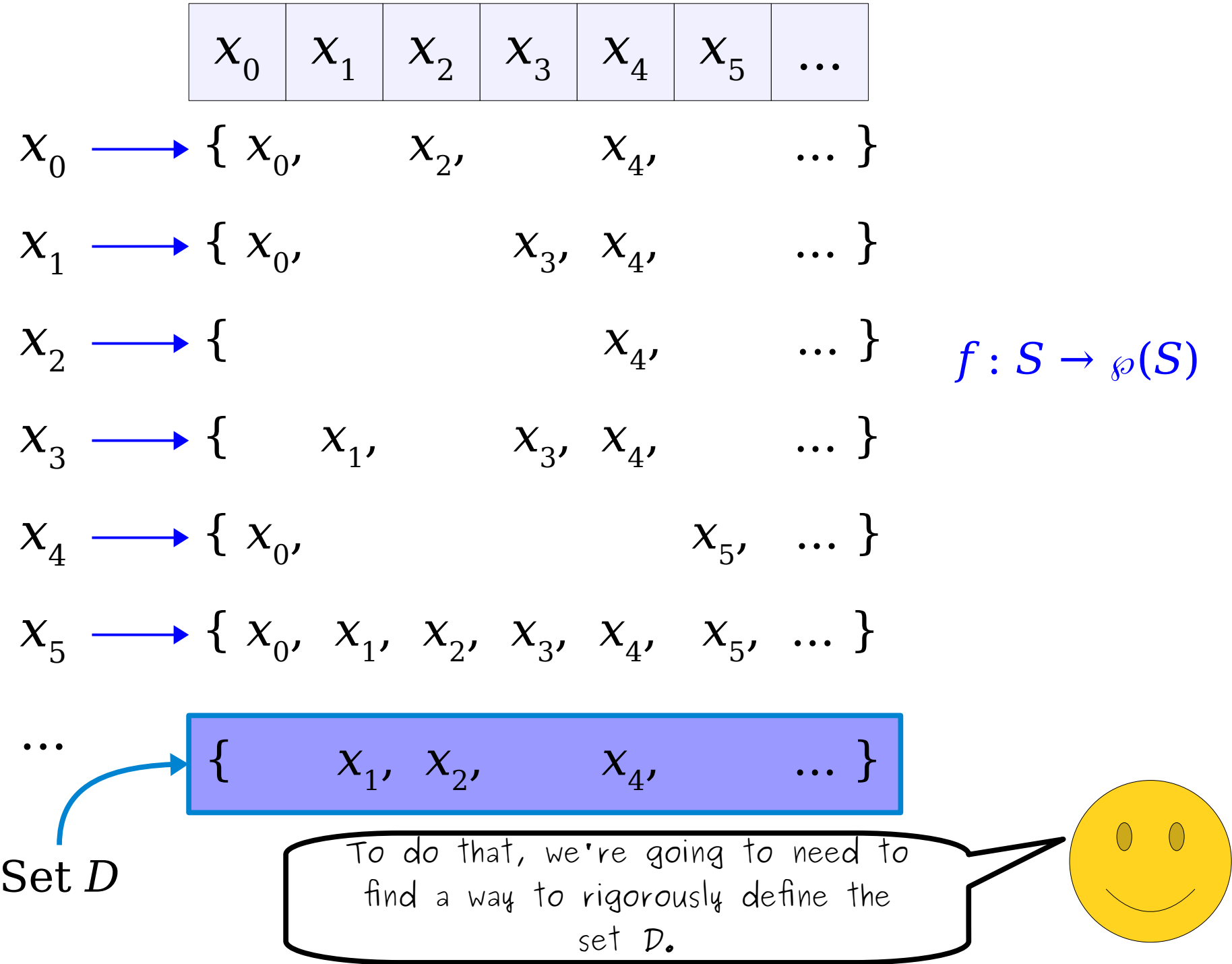
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



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Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$$x_0 \longrightarrow \{ x_0, \qquad x_2, \qquad x_4, \qquad \dots \}$$

$$x_1 \longrightarrow \{ x_0, \qquad \qquad \qquad x_3, \ x_4, \qquad \dots \}$$

$$x_2 \longrightarrow \{ \qquad \qquad \qquad \qquad \qquad x_4, \qquad \dots \}$$

$$x_3 \longrightarrow \{ \qquad x_1, \qquad \qquad \qquad x_3, \ x_4, \qquad \dots \}$$

$$x_4 \longrightarrow \{ x_0, \qquad \qquad \qquad \qquad \qquad x_5, \ \dots \}$$

$$x_5 \longrightarrow \{ x_0, \ x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ \dots \}$$

...

{

$x_1, \ x_2, \qquad \qquad x_4, \qquad \dots$

}

Set D

Once we've done that, we can write a formal proof that $f(x) \neq D$ for any $x \in S$.



$$f : S \rightarrow \wp(S)$$

Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

So... what is this set D ?



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Since sets are uniquely defined by the elements they contain, a reasonable first step is to see what's in D .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Although in this discussion I'm going to use the example f given here, this will work for any choice of f .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

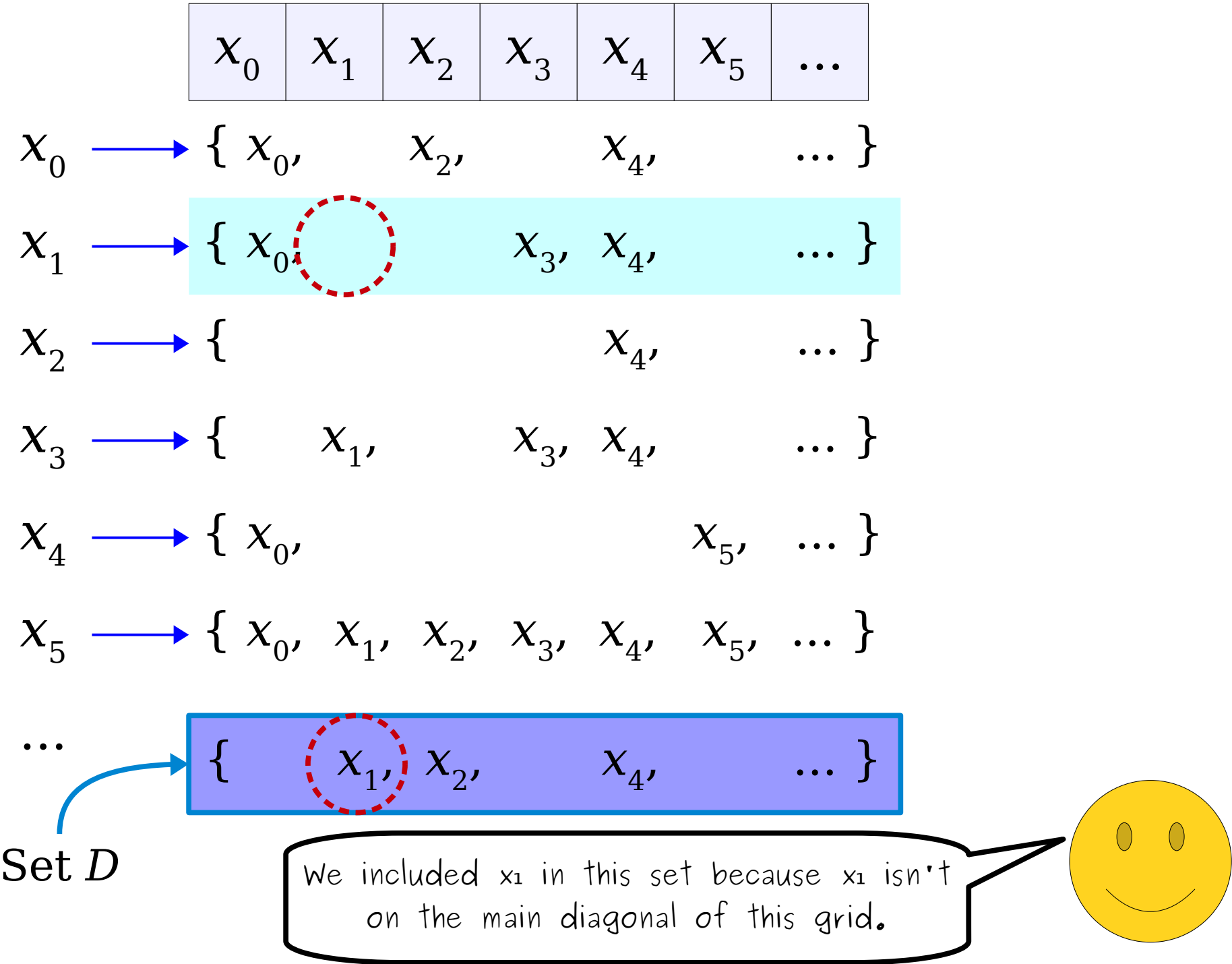
$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

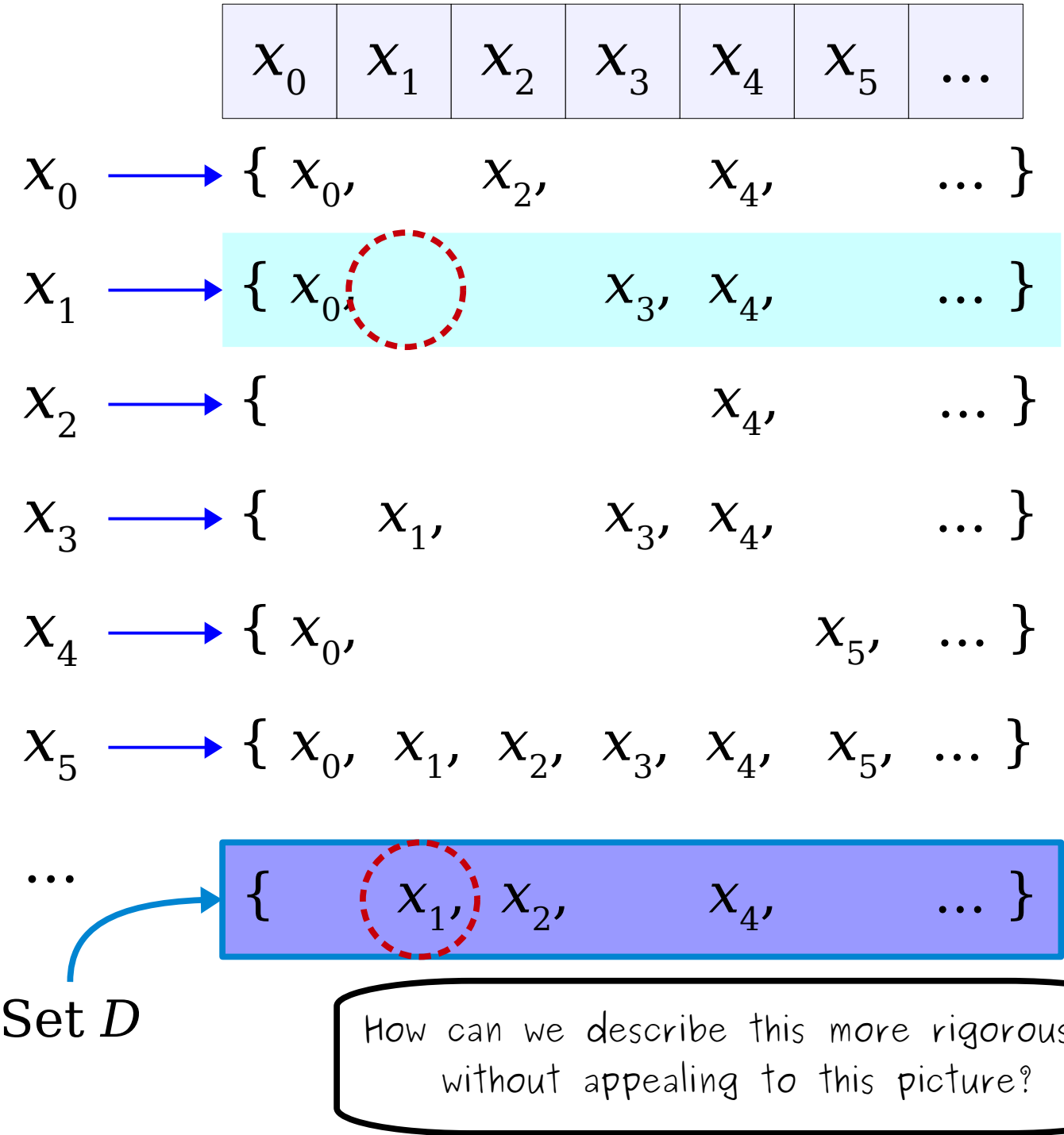
Let's look at x_1 , for example.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Well, let's take a look at that.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

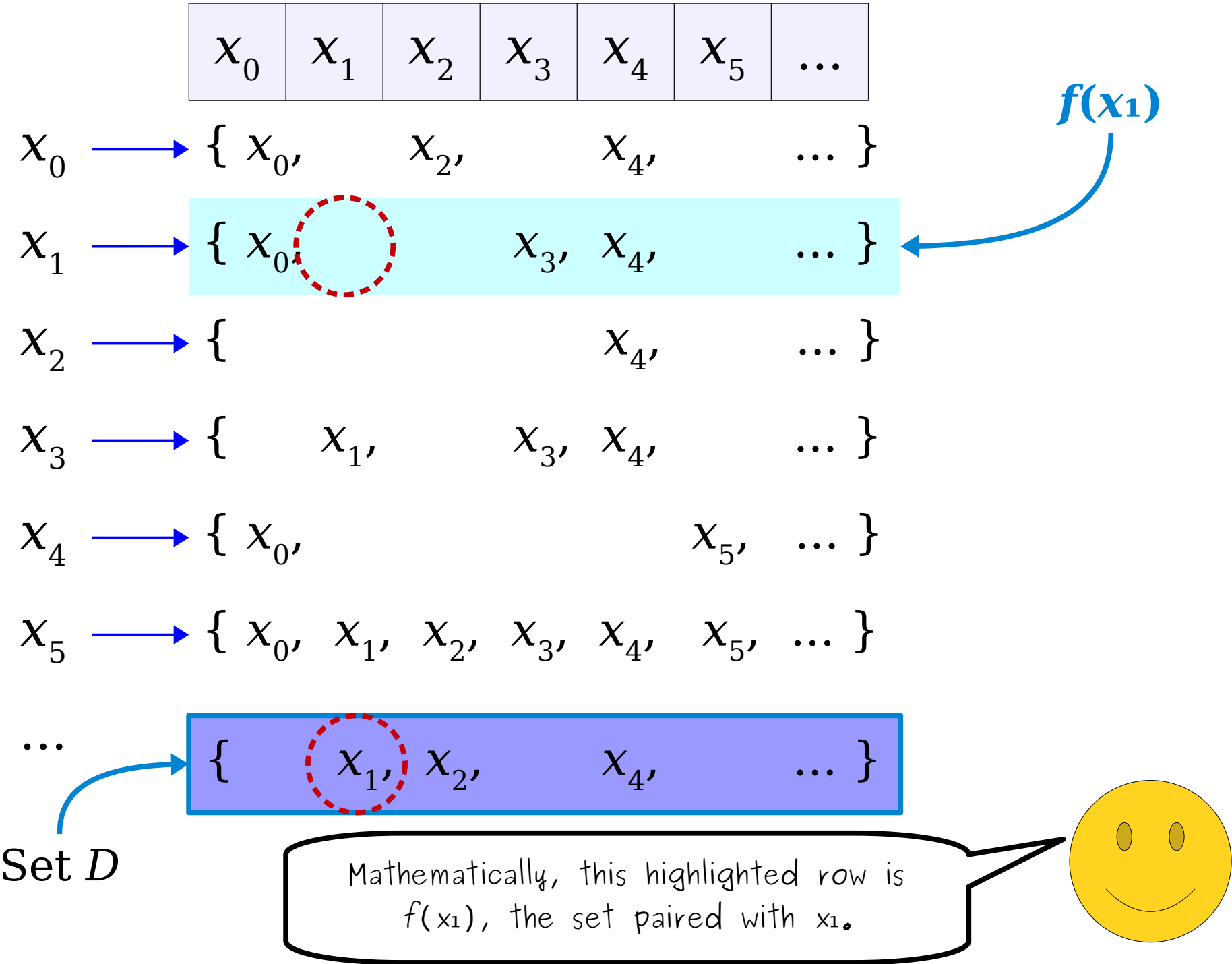
$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

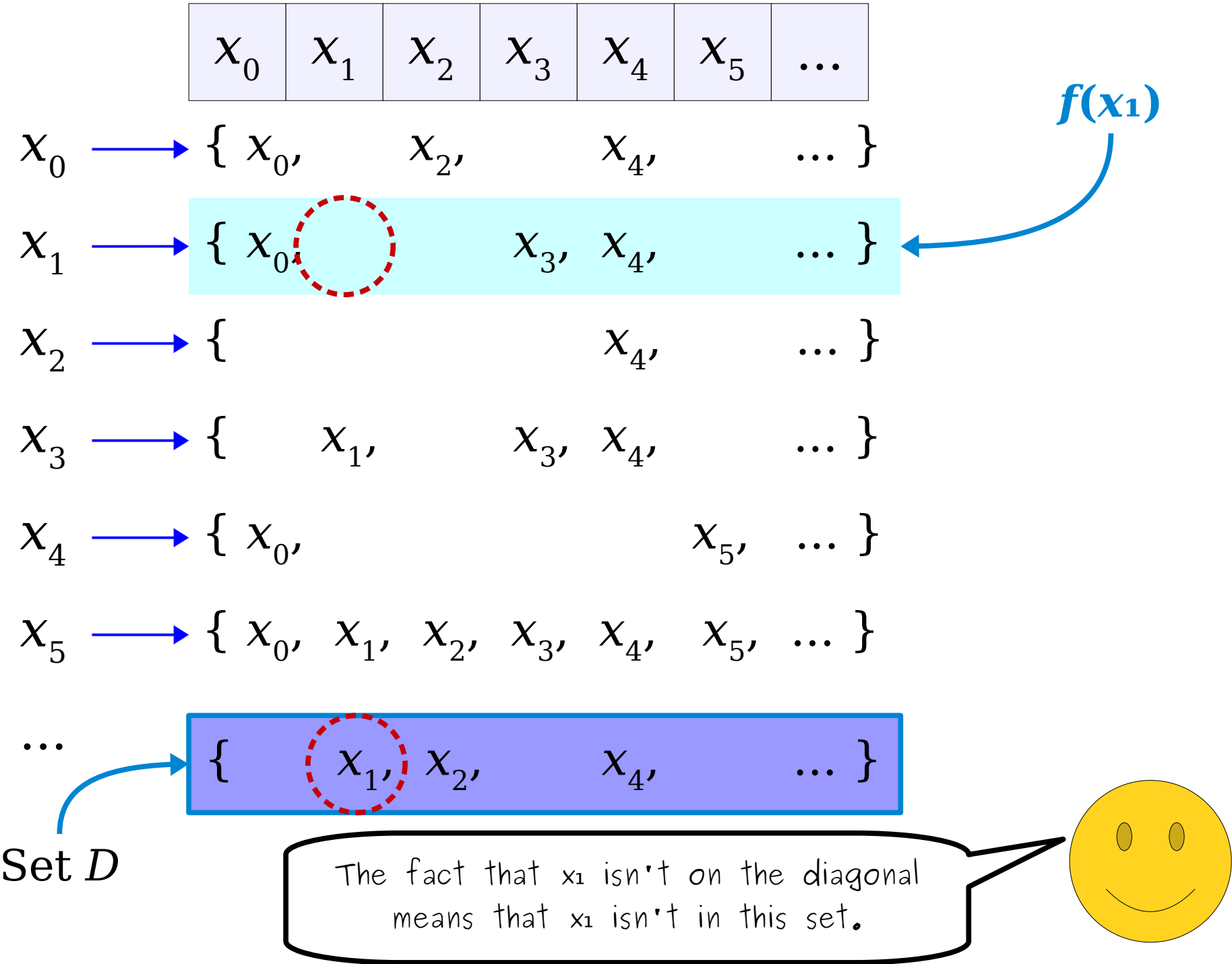
First, notice that we've highlighted this particular set. What set is this?



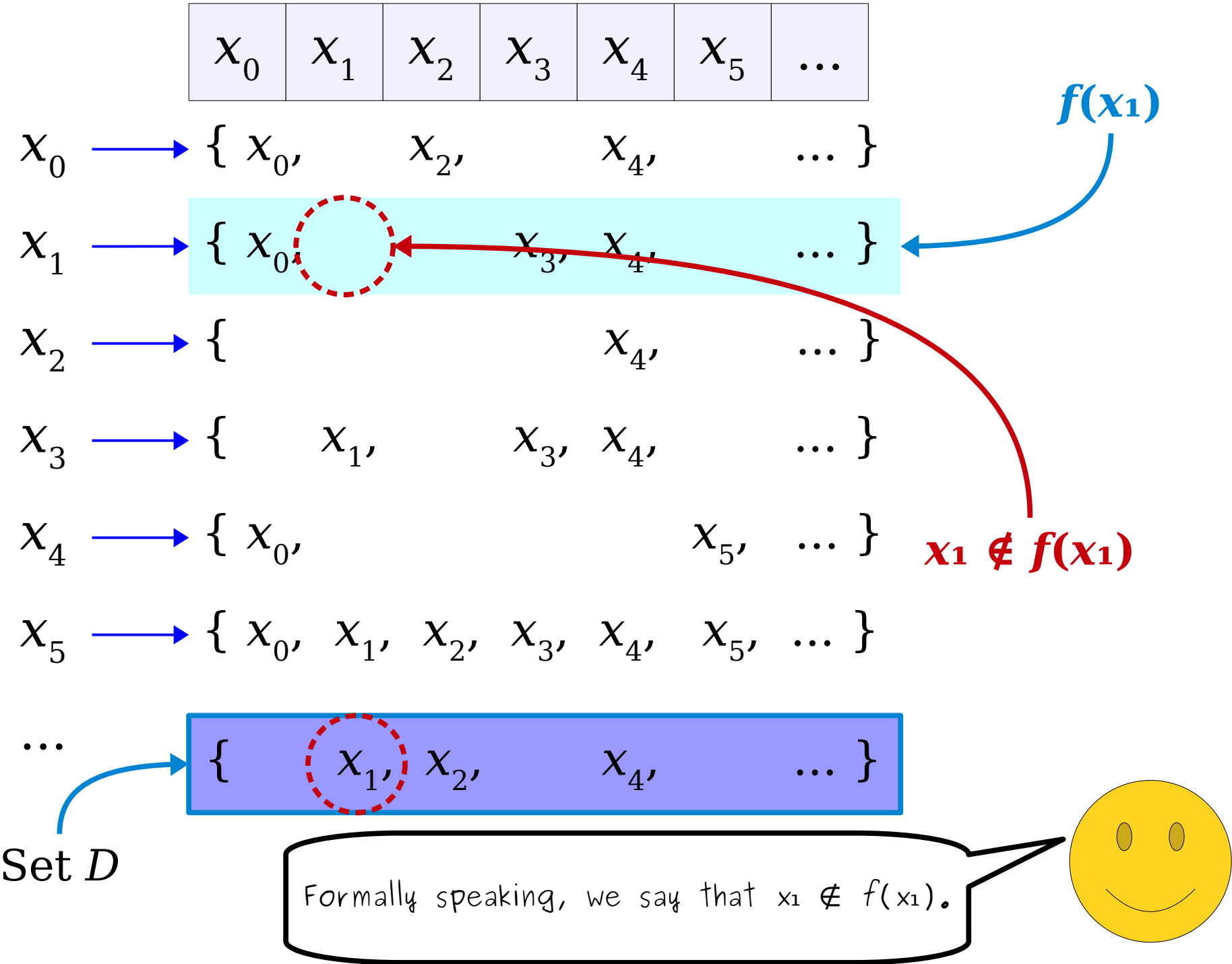
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



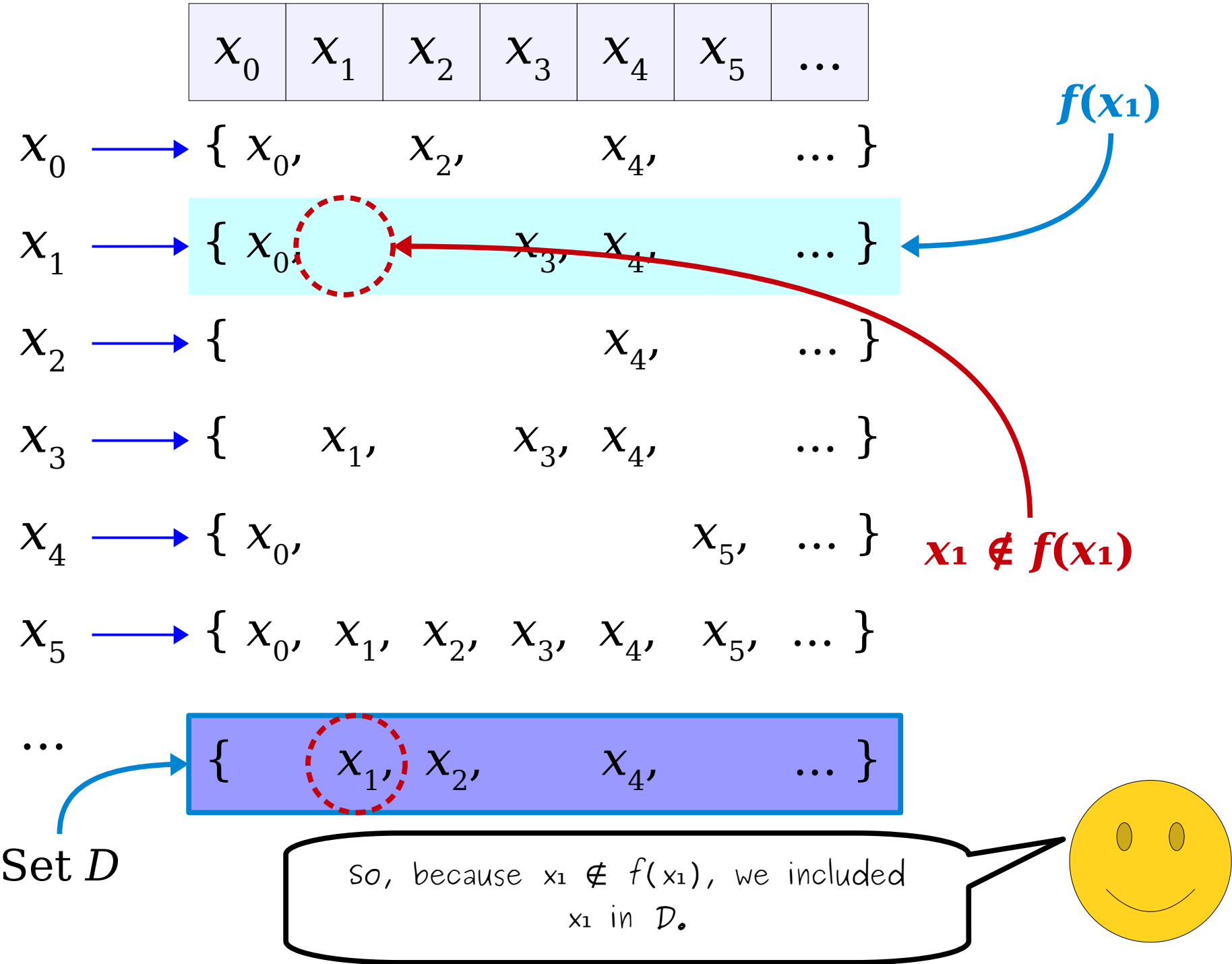
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



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Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

similarly, let's take a look at why x_2 is in our set.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad \quad \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

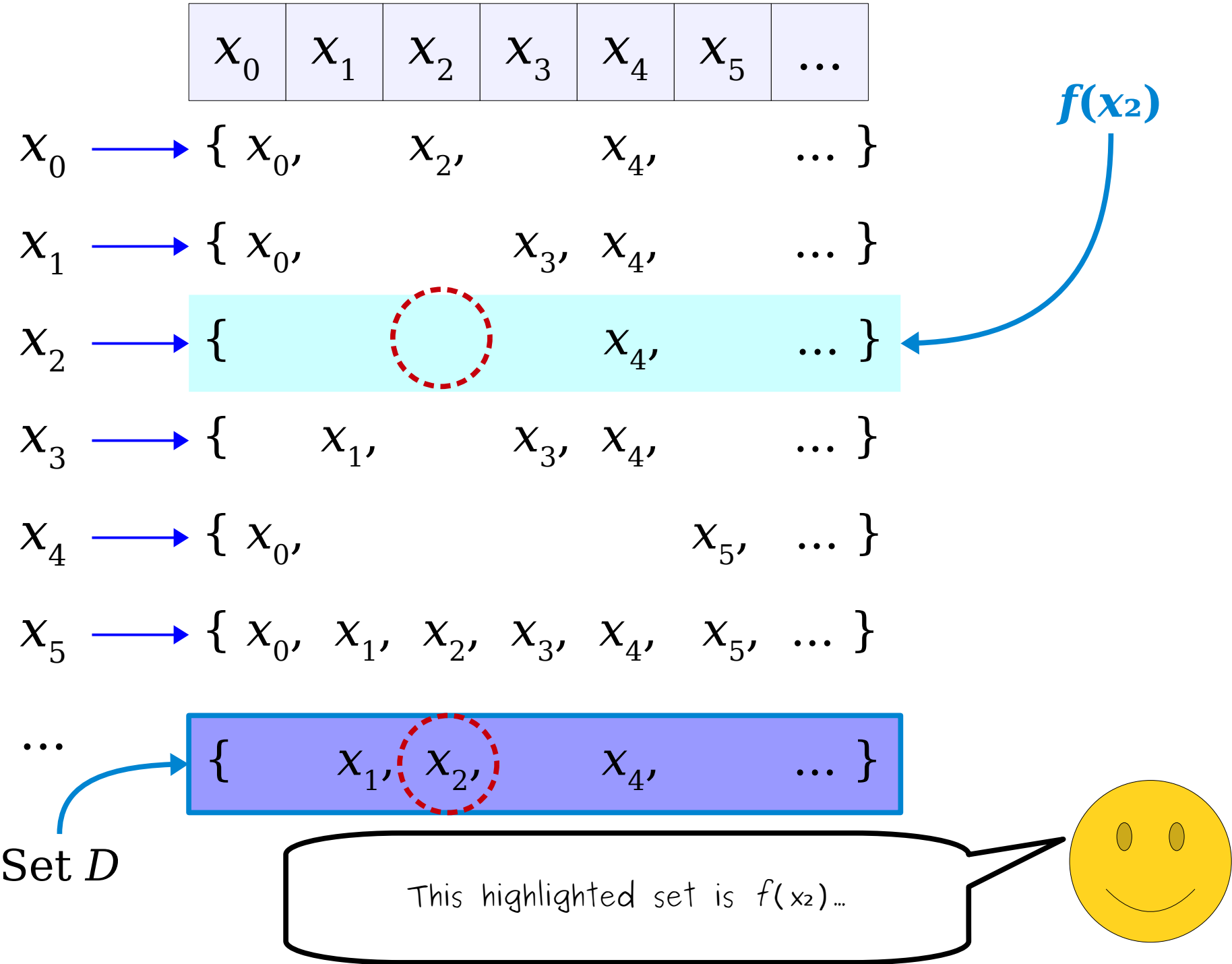
$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

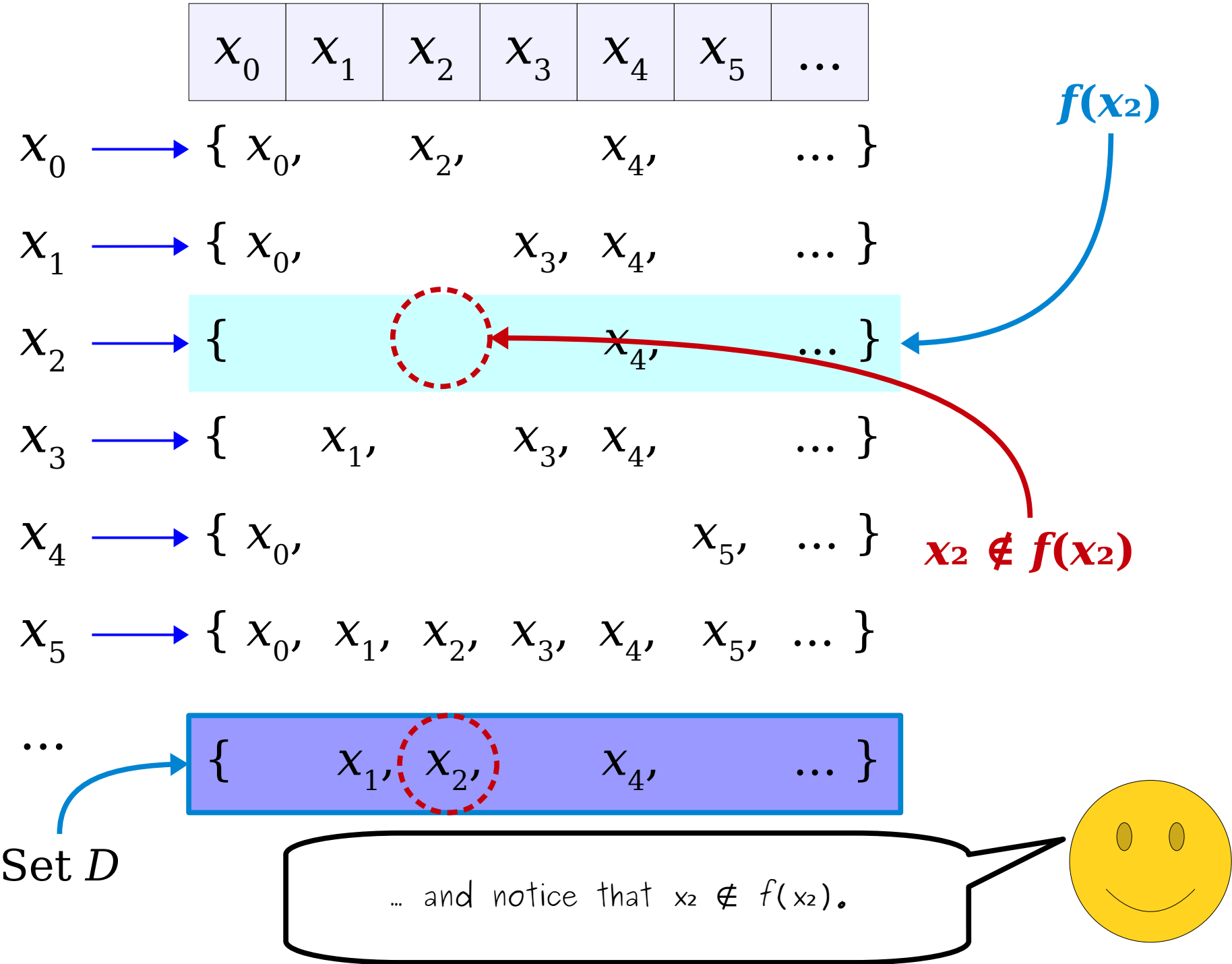
We included it because it's missing off the diagonal.



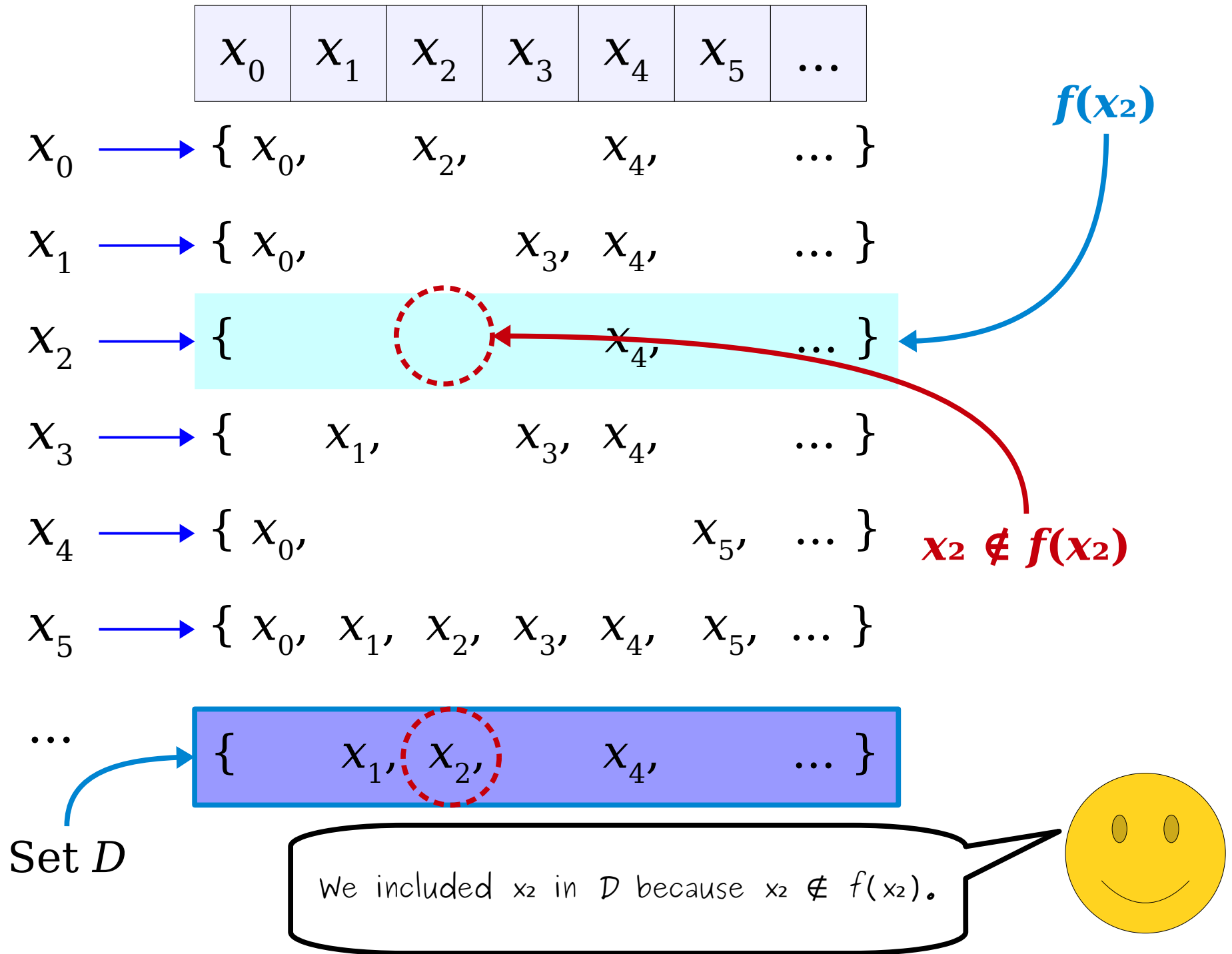
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



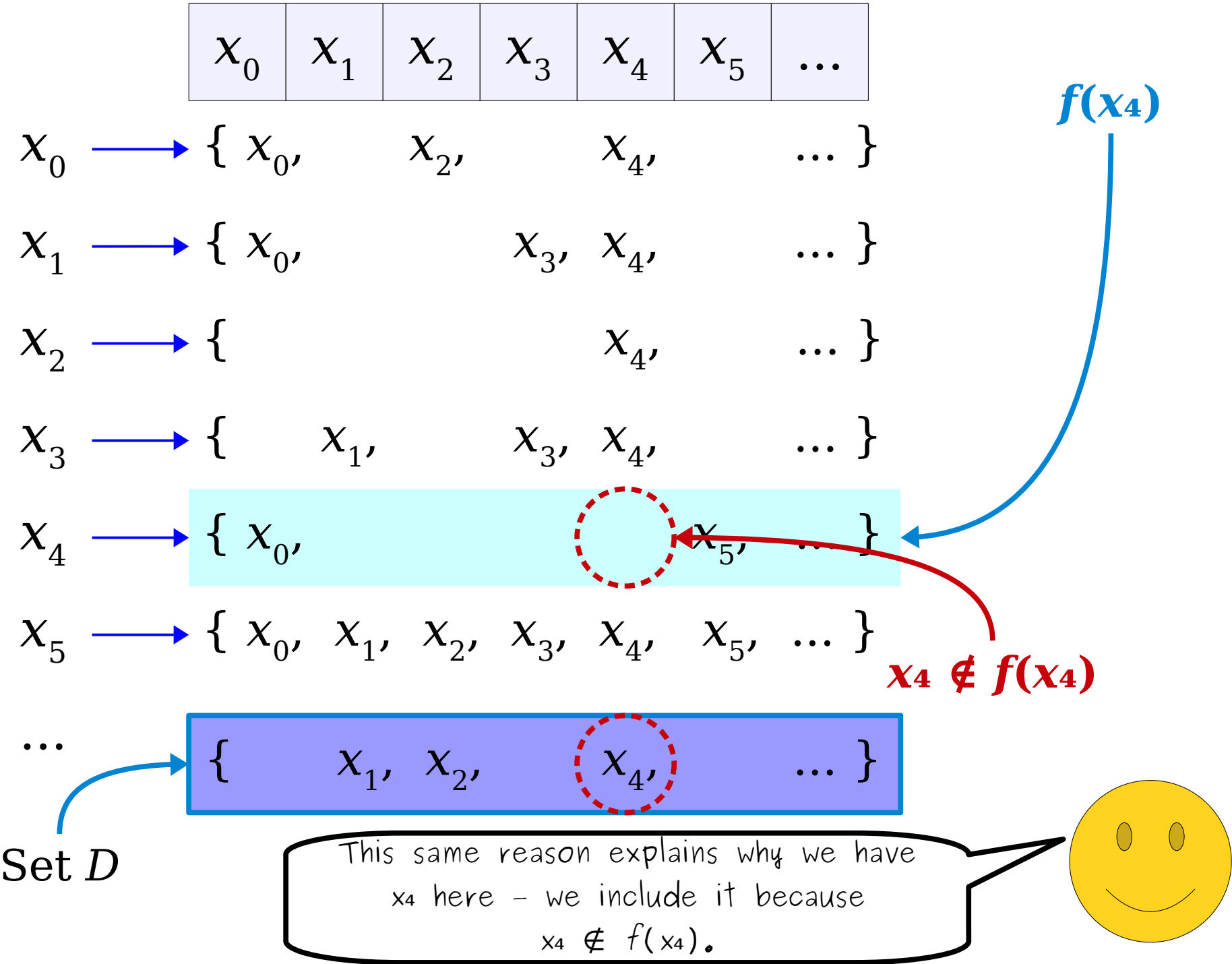
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Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

So it seems like we have a general pattern going here.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Specifically, if we find an x where $x \notin f(x)$, then it ends up in D .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

\mathbf{x}_0	\mathbf{x}_1	\mathbf{x}_2	\mathbf{x}_3	\mathbf{x}_4	\mathbf{x}_5	\dots
----------------	----------------	----------------	----------------	----------------	----------------	---------


$$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$$
$$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$$
$$x_2 \longrightarrow \{ x_4, \dots \}$$
$$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$$
$$x_4 \longrightarrow \{x_0,$$
$$x_5 \longrightarrow \{x_0, x_1, x_2, \dots, x_4, x_5\}$$


Diagram illustrating a set of features: $\{x_1, x_2, x_4, \dots\}$. An arrow points from an ellipsis (...) to the set notation.

Set D

specifically, if we find an x where $x \notin f(x)$, then it ends up in D .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

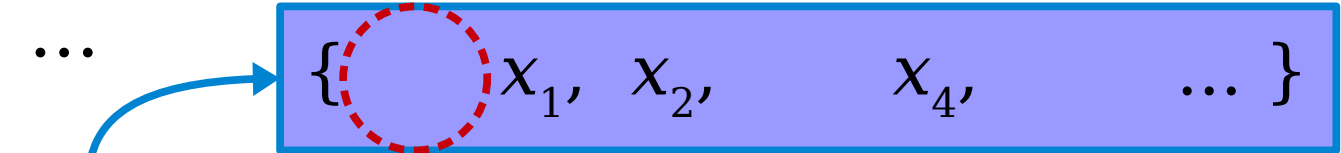
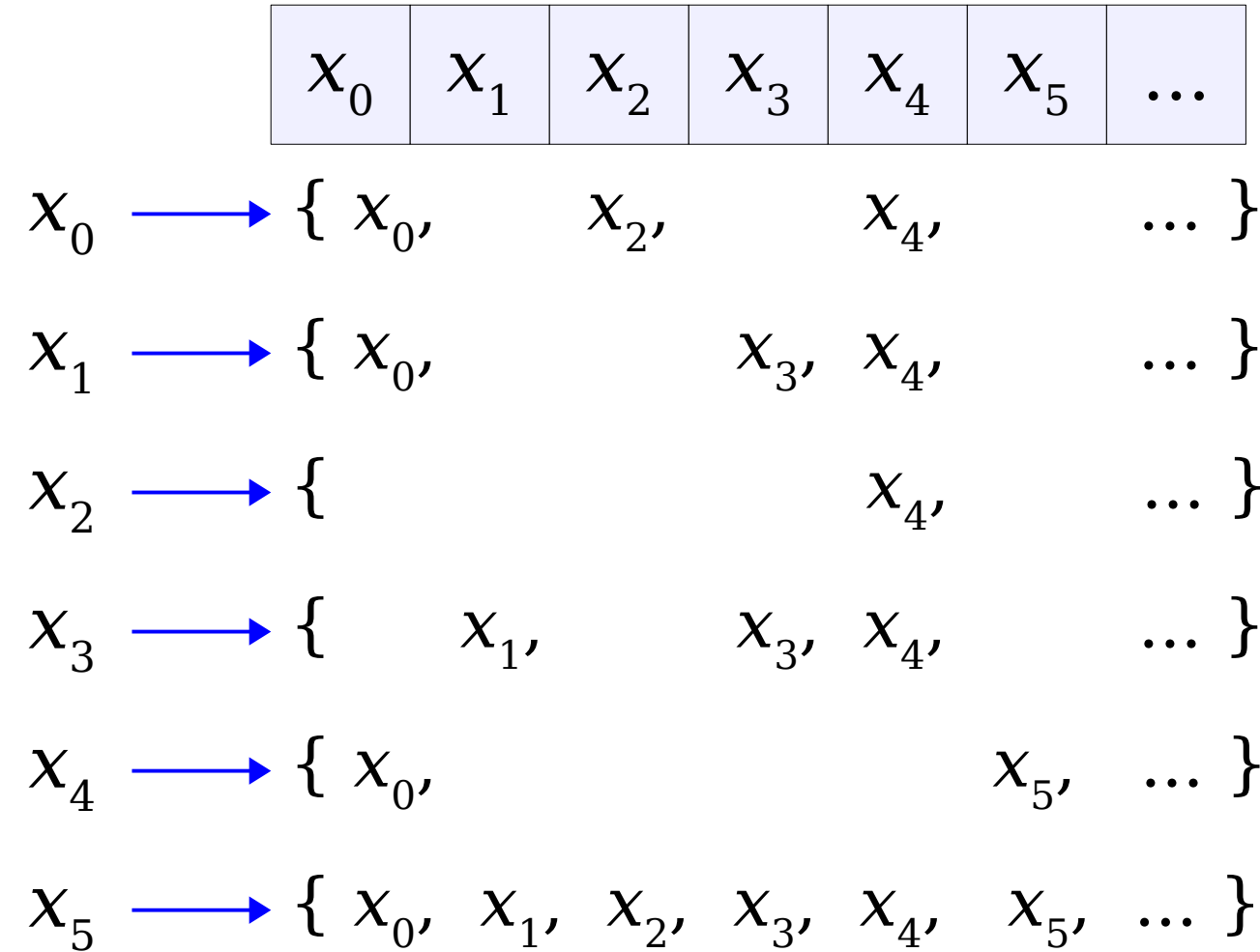
$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

So what about the elements that aren't in our set D ?



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

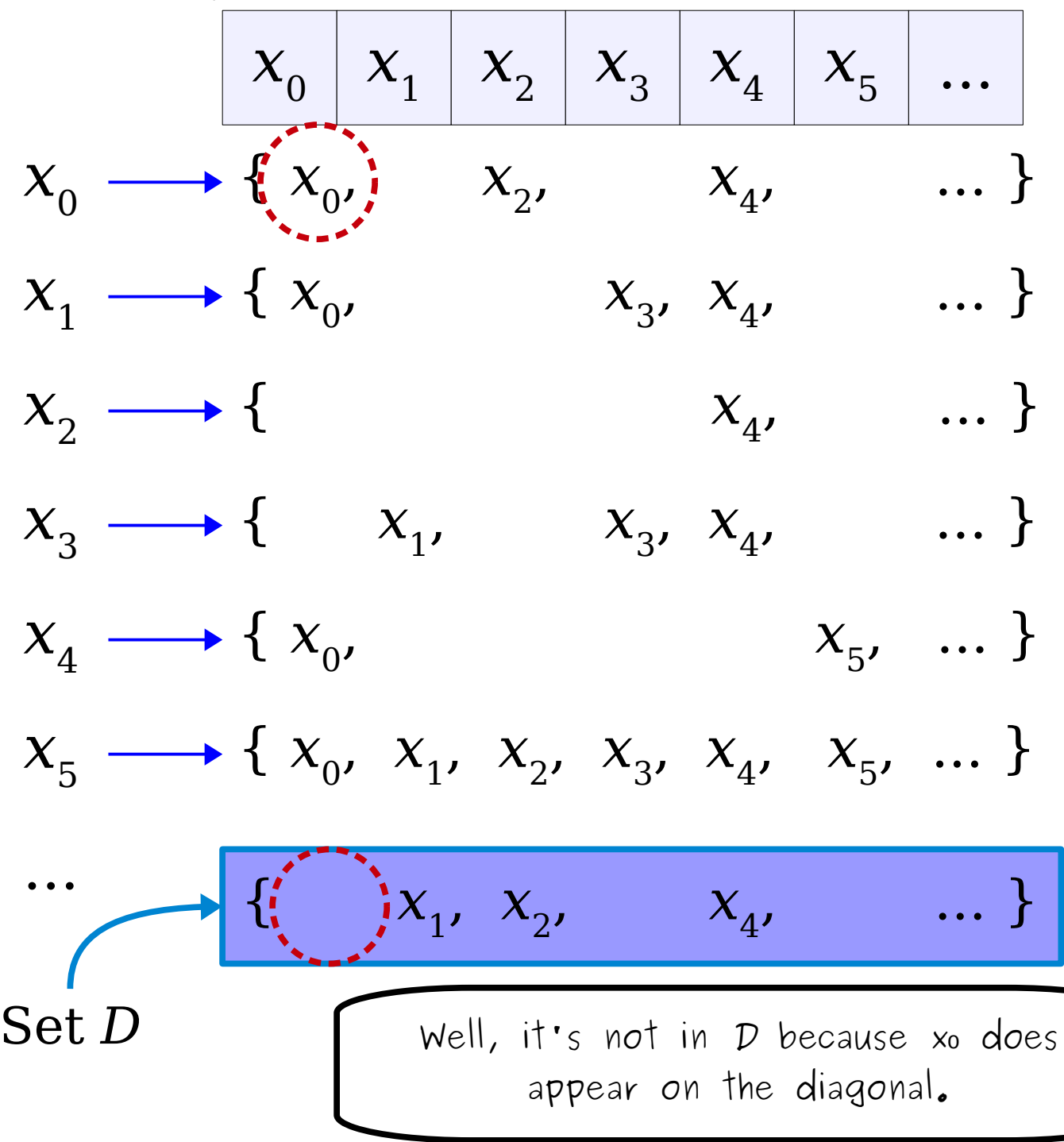


Set D

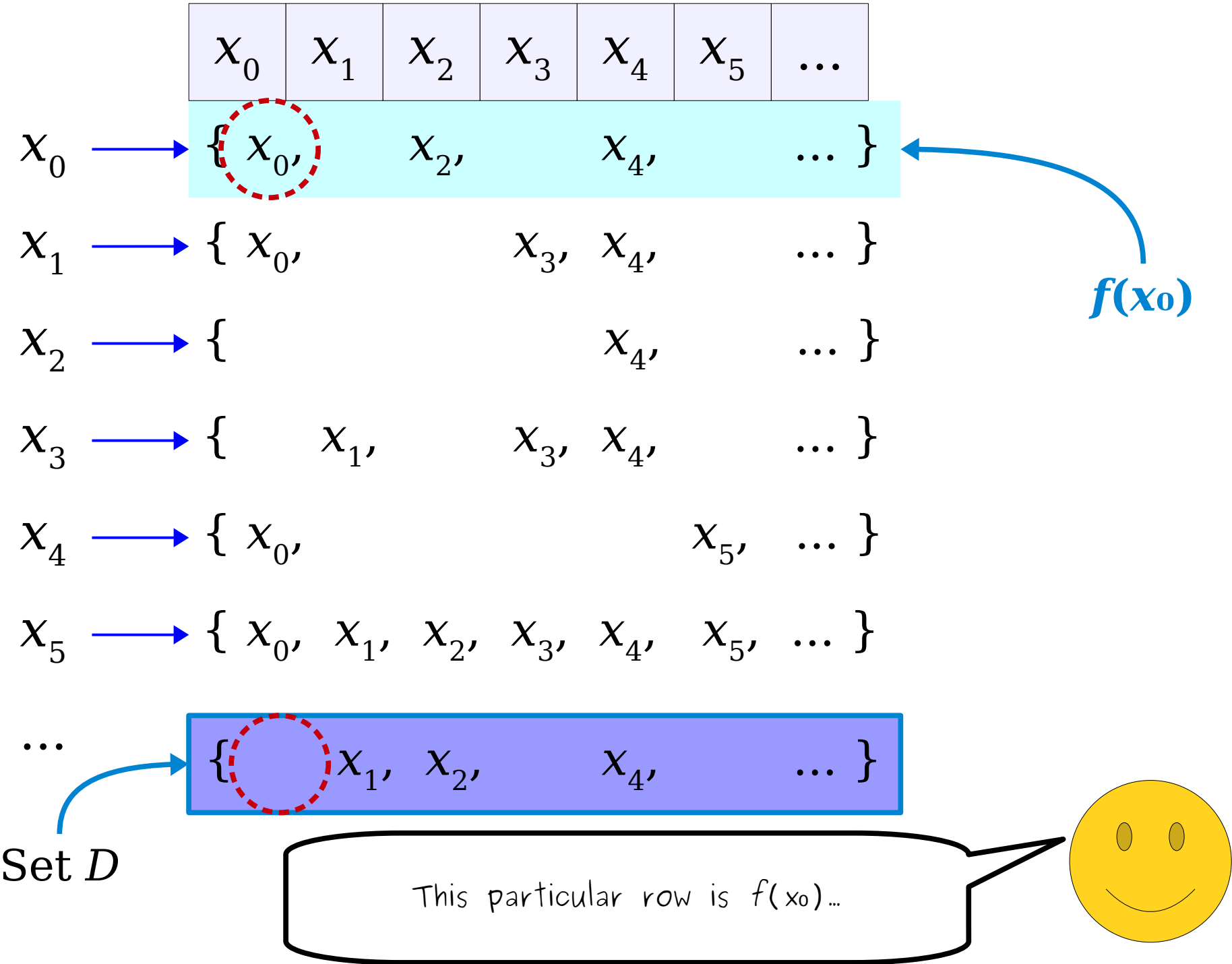
Let's start with x_0 . Why isn't it in D ?



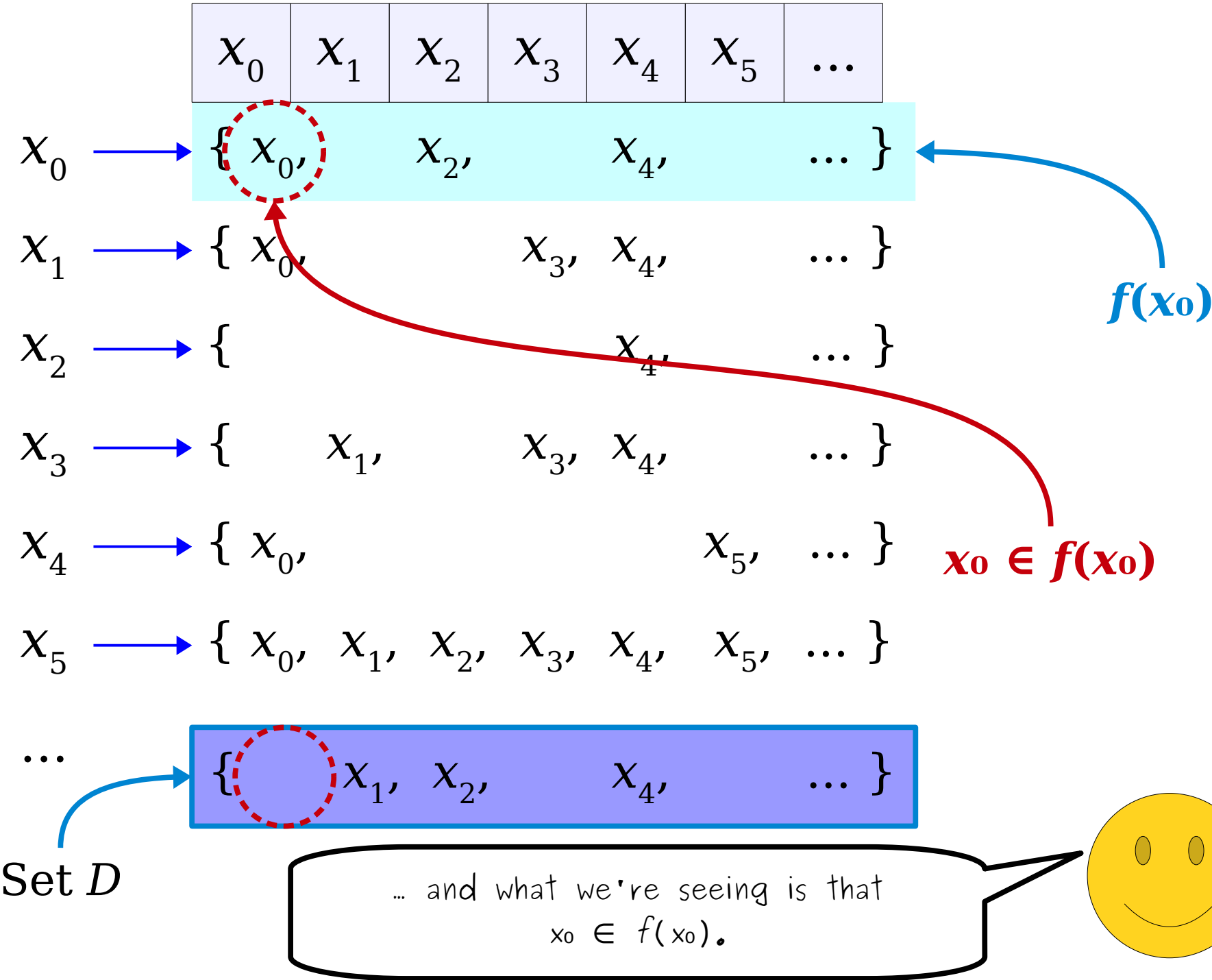
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



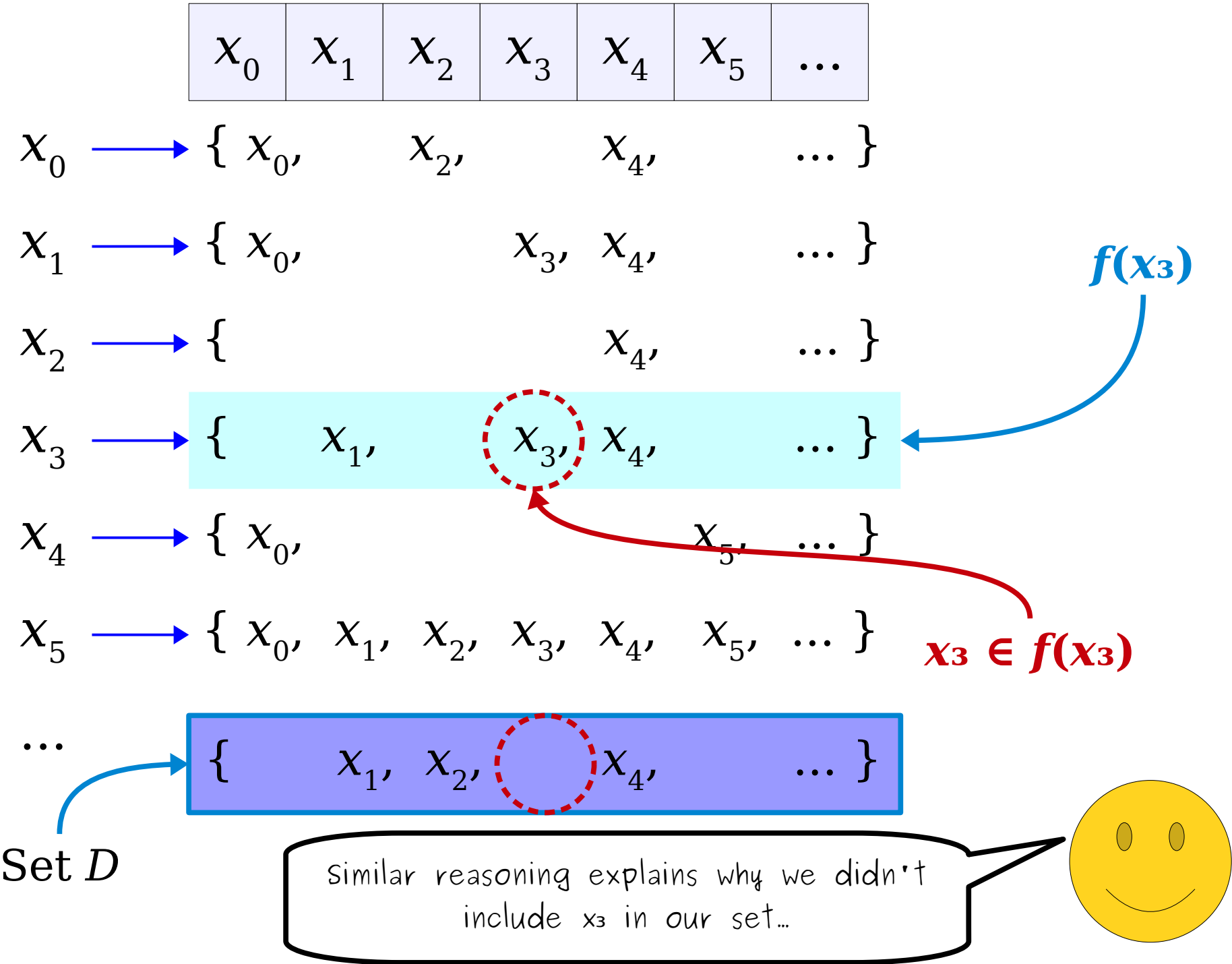
Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.



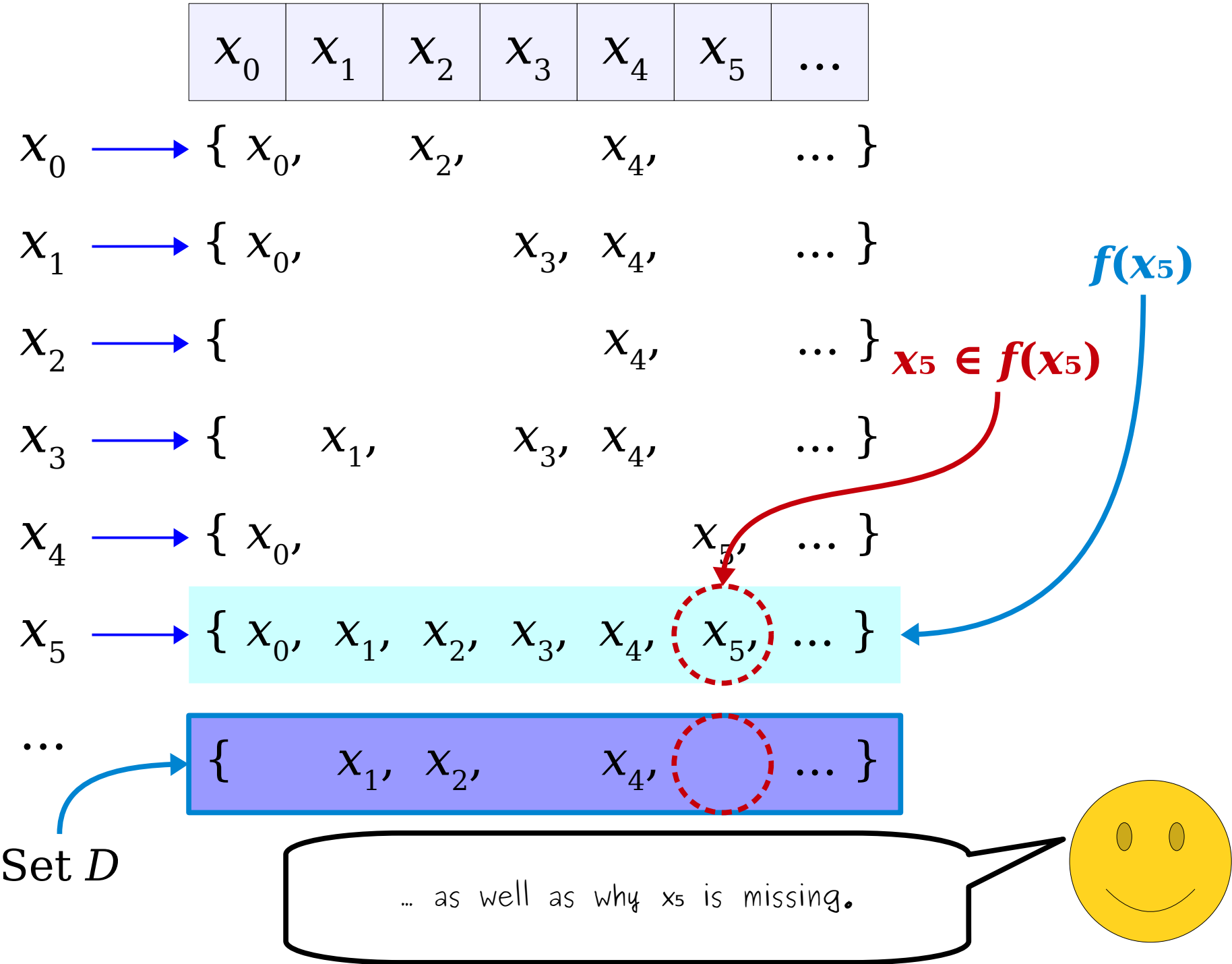
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x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

so it seems like we have another general rule...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

...the set D excludes every x where $x \in f(x)$.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

To recap...



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

If $x \notin f(x)$, then $x \in D$.

If we choose any x and see that $x \notin f(x)$, then $x \in D$.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$$

$$x_1 \longrightarrow \{ x_0, \quad \quad \quad x_3, \quad x_4, \quad \dots \}$$

$$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$$

$$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$$

$$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$$

$$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$$

$$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$$

Set D

If $x \notin f(x)$, then $x \in D$.
 If $x \in f(x)$, then $x \notin D$.

similarly, if we choose any x and find that $x \in f(x)$, then $x \notin D$.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	\dots
-------	-------	-------	-------	-------	-------	---------

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

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$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots \longrightarrow \{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

Collectively, this lets us define our set D purely mathematically.



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

\mathbf{x}_0	\mathbf{x}_1	\mathbf{x}_2	\mathbf{x}_3	\mathbf{x}_4	\mathbf{x}_5	\dots
----------------	----------------	----------------	----------------	----------------	----------------	---------


$$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$$
$$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$$
$$x_2 \longrightarrow \{ x_4, \dots \}$$
$$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4,$$
$$x_4 \longrightarrow \{x_0,$$
$$x_5 \longrightarrow \{x_0, x_1, x_2, x_3, x_4, x_5\}$$


Diagram illustrating a set of features: $\{x_1, x_2, x_4, \dots\}$. An arrow points from an ellipsis (...) to the set notation.

Set D

specifically:

$$\mathcal{D} = \{ x \in S \mid x \notin f(x) \}$$

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{ x \in S \mid x \notin f(x) \}$$



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

If $x \notin f(x)$, then $x \in D$.
 If $x \in f(x)$, then $x \notin D$.

$D = \{ x \in S \mid x \notin f(x) \}$

... $\{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

Take a minute to see why this is – it's a consequence of the two above rules.

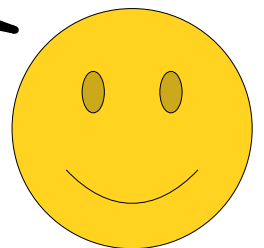


Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{x \in S \mid x \notin f(x)\}$$

Phew! That was a long setup.

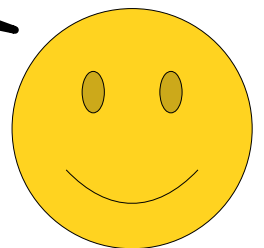


Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{x \in S \mid x \notin f(x)\}$$

The whole point of that visual exercise
was to figure out a mathematical way
of describing that diagonal set.

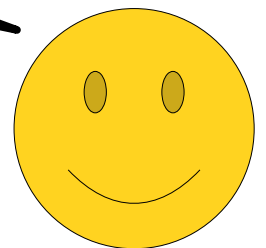


Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{x \in S \mid x \notin f(x)\}$$

At this point, we actually don't need to draw any more pictures. Everything we need is summarized by the above symbolic notation.

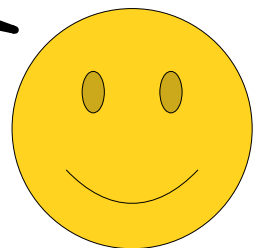


Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$$D = \{x \in S \mid x \notin f(x)\}$$

In fact, just knowing this particular choice
of D allows us to write a formal proof
that $|S| \neq |\wp(S)|$!

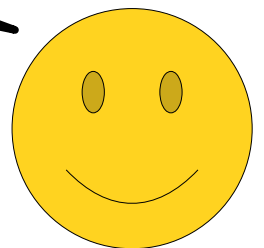


Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

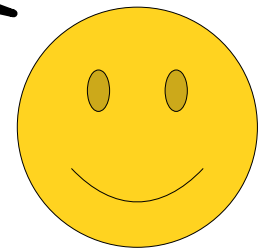
$$D = \{x \in S \mid x \notin f(x)\}$$

Let's see what it looks like.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

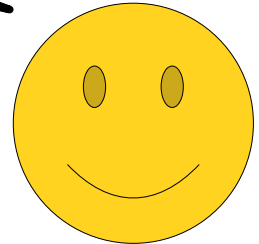
Okay! 3... 2... 1... let's jam!



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

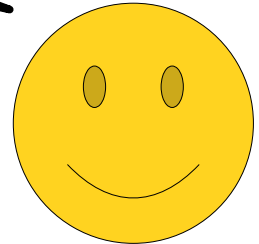
This proof starts off, as many proofs do, by just saying what we're going to be doing.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

This is mostly a recap of our previous strategy.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

This is where we lay out the specific strategy we're going to use. There's a lot of ways to prove that there aren't any bijections, and we're going to do it by (as you saw) showing that no function from S to $\wp(S)$ can be a bijection.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

This is where we lay out the key idea behind the proof – we can start with this arbitrary function f and produce a set D that f doesn't map anything to.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We worked really hard to come up with this set D , and now we get to see just how cool a set it is.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

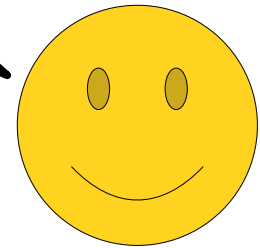
Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$.

Ultimately, we're going to show that f isn't surjective. Here, we're saying how: we're going to show that nothing maps to D .



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$.

(I chose the name y here because the name x was taken earlier, and I didn't want to confuse the two. You'll see why in a second.)



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$.

And now things really get rolling. Let's suppose that you could find a choice of y such that $f(y) = D$.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

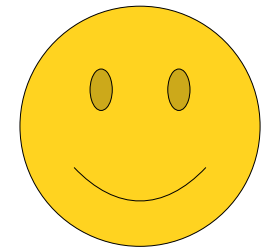
Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$.

Our intuition behind building the set D was to get a contradiction between whether $y \in f(y)$ and whether $y \in D$. Specifically,
if $y \notin f(y)$, then $y \in D$, and
if $y \in f(y)$, then $y \notin D$.

This is just the mathematical way of talking about what happens on the diagonal.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$.

The rest of this proof is a way of formalizing, mathematically, what goes wrong.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

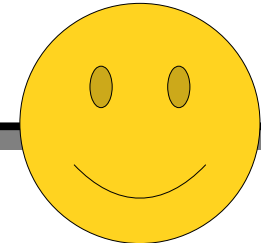
Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of D , we know that

$$y \in D \text{ if and only if } y \notin f(y). \quad (2)$$

We're going to begin with this observation.
Let's see where this comes from.



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that f is not surjective.

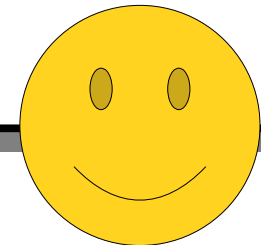
Starting with f , we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \quad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of D , we know that

$$y \in D \text{ if and only if } y \notin f(y). \quad (2)$$

What would have to happen for
 $y \in D$ to be true?



Lemma 2: If S is a set, then $|S| \neq |\wp(S)|$.

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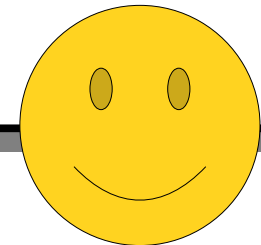
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Well, according to the definition of D ...



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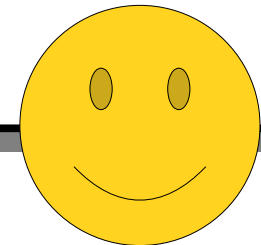
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... that would mean that y isn't an element of the set it belongs to.



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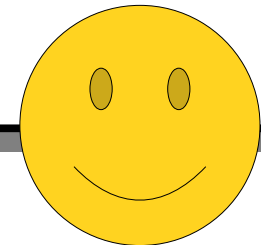
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We can put an "iff" between these two statements because D contains precisely the elements x where $x \notin f(x)$.



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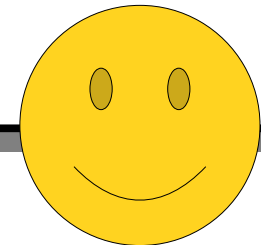
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Right now, nothing special seems to happen. Here's where things get interesting.



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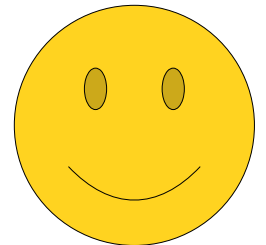
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By assumption, $f(y) = D$. Combined with (2), this tells us

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We specifically assumed that $f(y) = D$.
That means we can rewrite statement (2)
as shown here. If you're wondering
why we can do this...



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... notice that we're just substituting D for $f(y)$ in the right-hand side of this statement.



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And hey! Does this statement look, you know, kinda fishy?



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It should! This says that a statement
is true if and only if it's false...
but that's impossible!



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This is impossible.

Hey! That's what I just said.
Now we just need to bring it home.



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This is impossible. We have reached a contradiction, so our assumption must have been wrong.

This contradiction means, as most do,
that our initial assumption was wrong.



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Scanning back in the proof, we can see that this is what we were assuming – that there was something that mapped to D .



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This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so f is not surjective.

So I guess that means that's not possible,
and that in turn means that f is not
surjective!



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And at this point we're basically done! We showed that an arbitrary function f from S to $\wp(S)$ isn't a bijection, so we can conclude that $|S| \neq |\wp(S)|$.



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This is a dense proof. There's a lot of just plain unpacking what we need to prove, plus the highly unusual choice of D .



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My hope is that you'll be able to wrap your head around this as a result of two factors.



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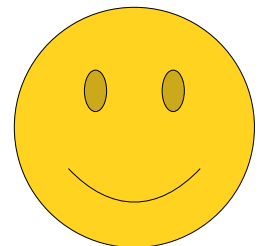
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First, you have the visual intuition for the proof. I encourage you to look back at the visual and to make sure you see where these choices come from.



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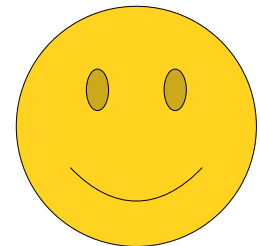
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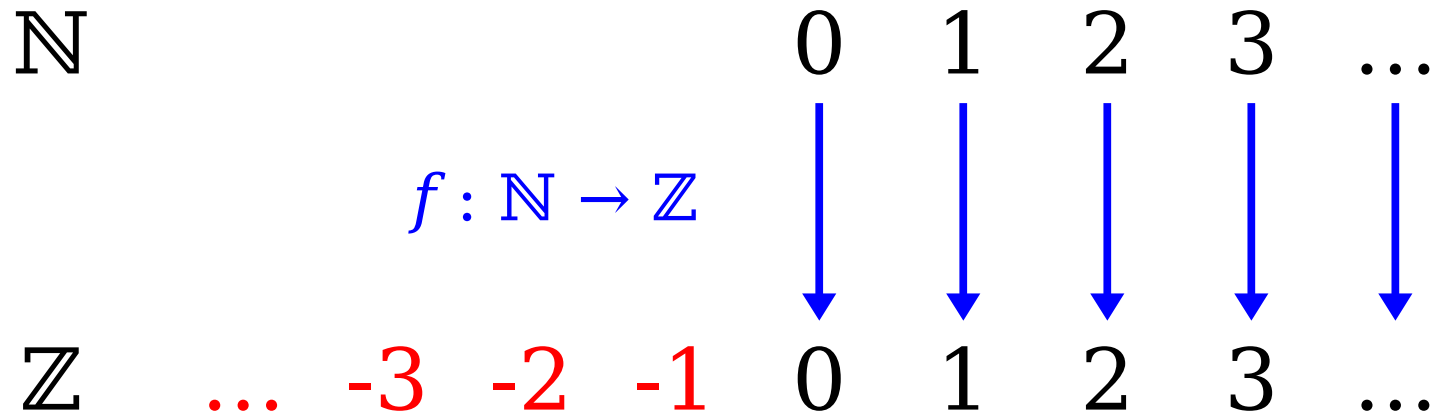
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Second, you'll have the upcoming problem set, where you'll get to play around with the proof to see how it works. Trust us, it's not as bad as it looks!



Phew! We've covered a lot of ground here. Let's take stock of what we covered here.





$|S| \leq |T|$ if there is an **injective** function $f: S \rightarrow T$

First, we came up with a rigorous definition of what $|S| \leq |T|$ means. This allows us to prove results about how set cardinalities rank against one another... something you (hypothetically speaking) might need for the upcoming problem set.



$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective
function $f : S \rightarrow T$

No function $f : S \rightarrow T$
is a bijection.

Second, we used that definition to come
up with a definition for $|S| < |T|$, which
lets us say that one set is strictly smaller
than another.



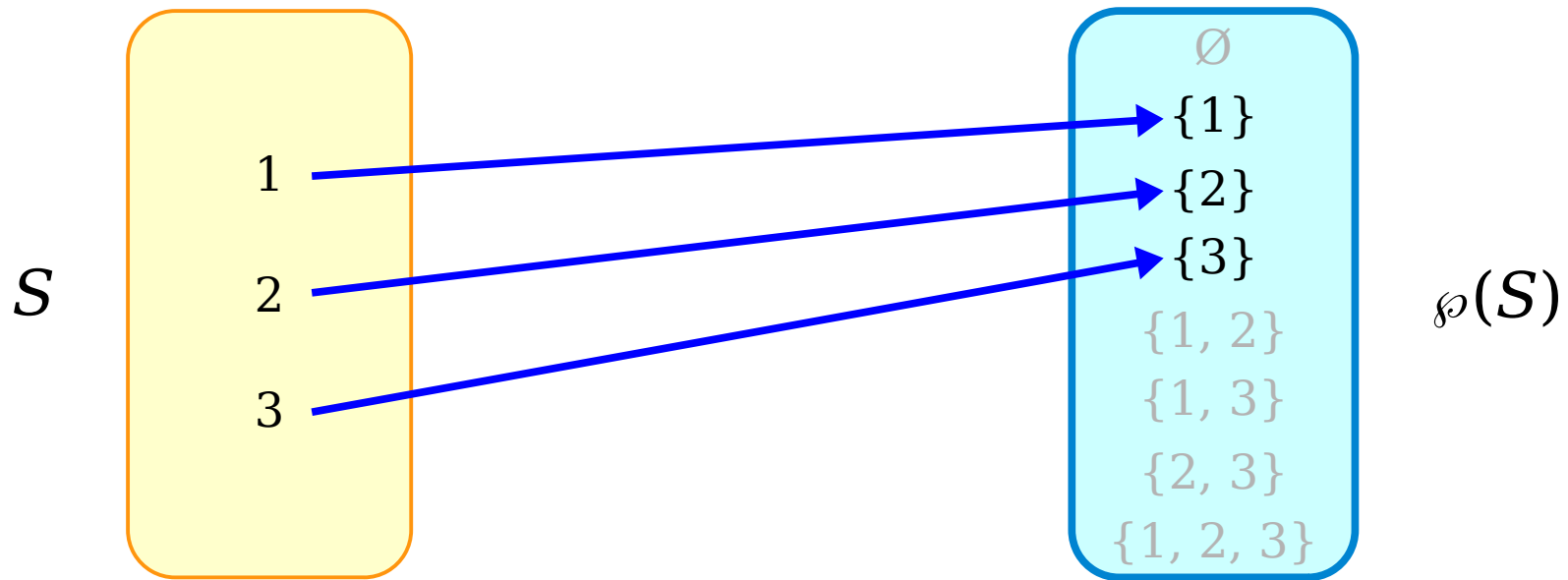
There is an injection $f: S \rightarrow \wp(S)$
There is *no* bijection $f: S \rightarrow \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

This in turn allowed us to determine what, specifically, we need to prove to prove Cantor's theorem.



Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f: S \rightarrow \wp(S)$$
$$f(x) = \{x\}$$

We saw how to find an injection from S to $\wp(S)$ for any set S .



Lemma 2: If $f : S \rightarrow \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	...
-------	-------	-------	-------	-------	-------	-----

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad \quad \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad \quad \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

If $x \notin f(x)$, then $x \in D$.
 If $x \in f(x)$, then $x \notin D$.

$D = \{ x \in S \mid x \notin f(x) \}$

... $\{ \quad x_1, \quad x_2, \quad \quad x_4, \quad \dots \}$

Set D

We used the visual intuition of the diagonal argument to precisely define the diagonal set.



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Finally, we wrote a formal proof that
 $|S| \neq |\wp(S)|$.



Hope this helps!

Please feel free to ask
questions if you have them.



Did you find this useful? If so, let us know! We can go and make more guides like these.

