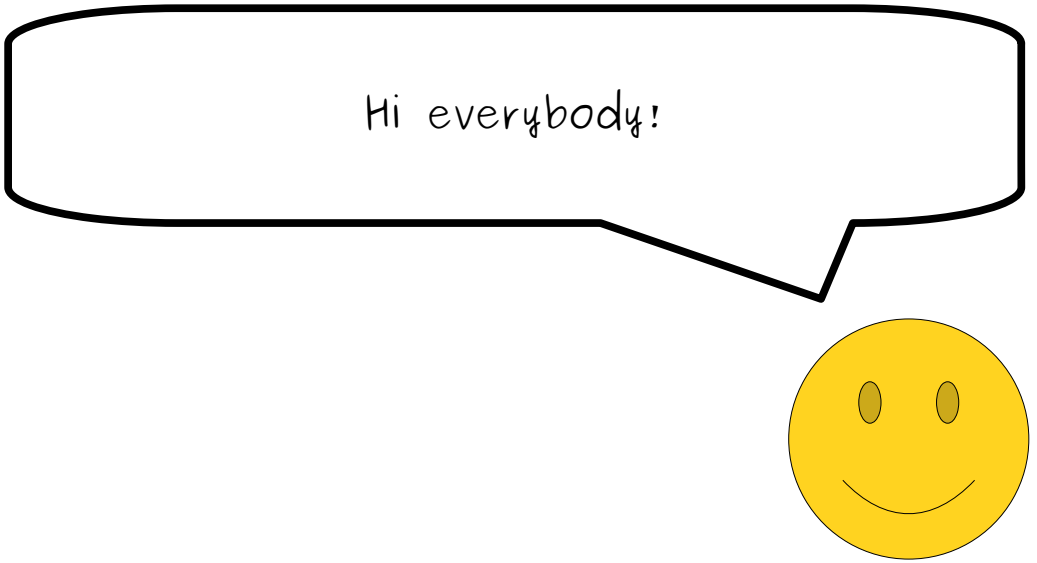
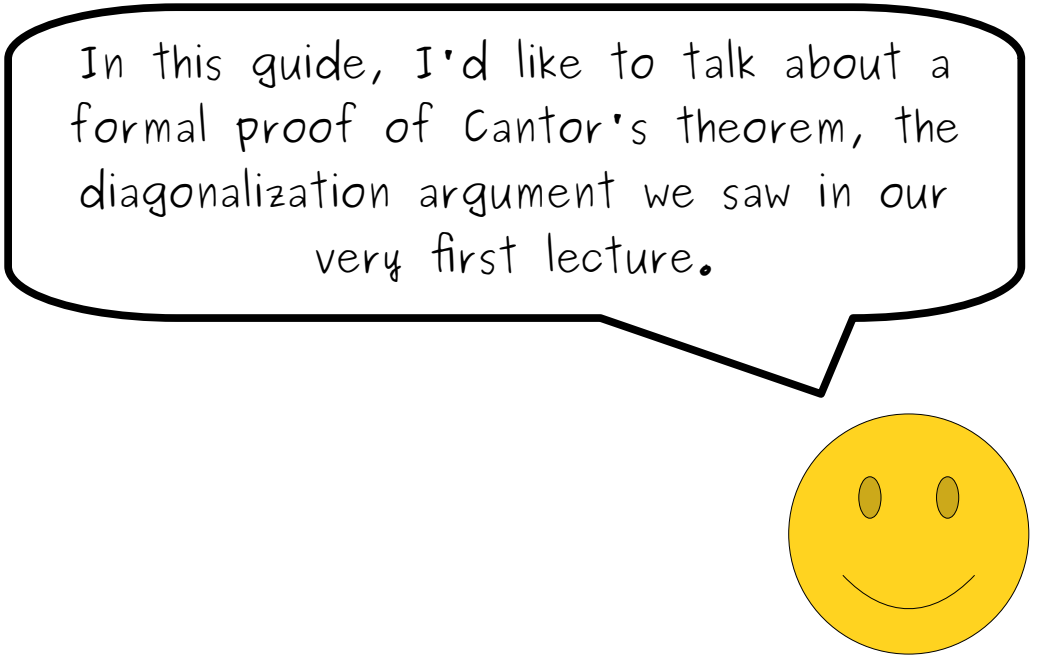
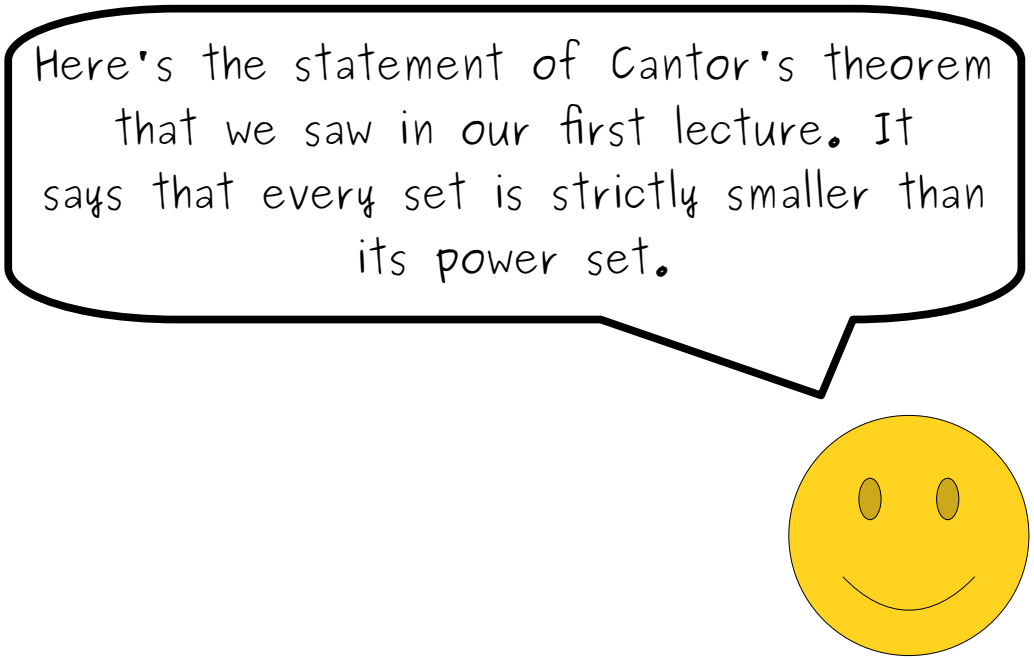
Guide to Cantor's Theorem


Hi everybody!


In this guide, I'd like to talk about a formal proof of Cantor's theorem, the diagonalization argument we saw in our very first lecture.

If S is a set, then $|S| < |\wp(S)|$



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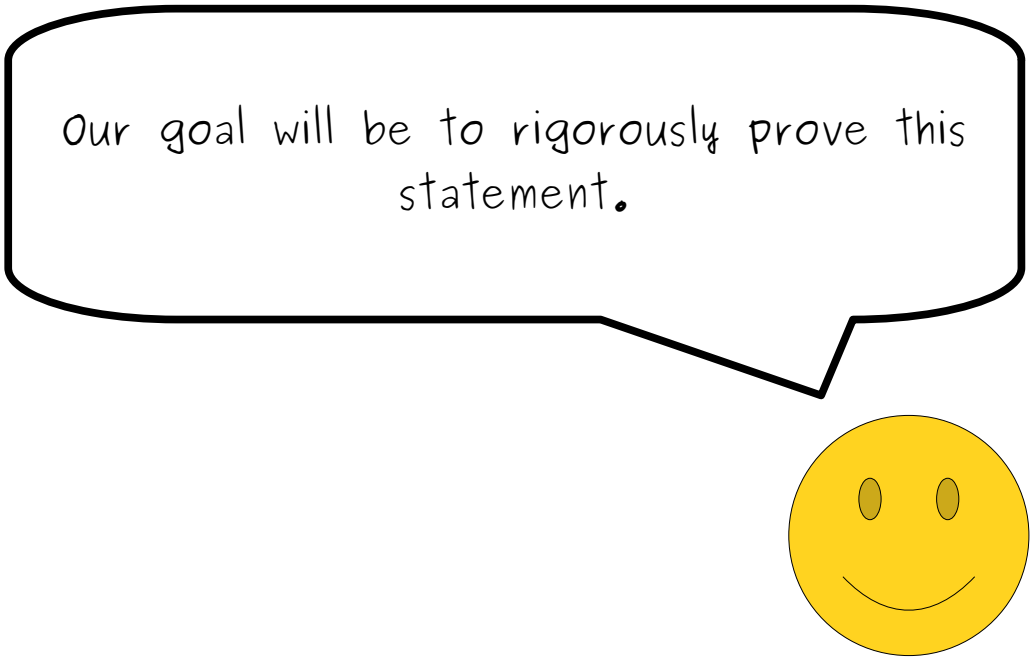

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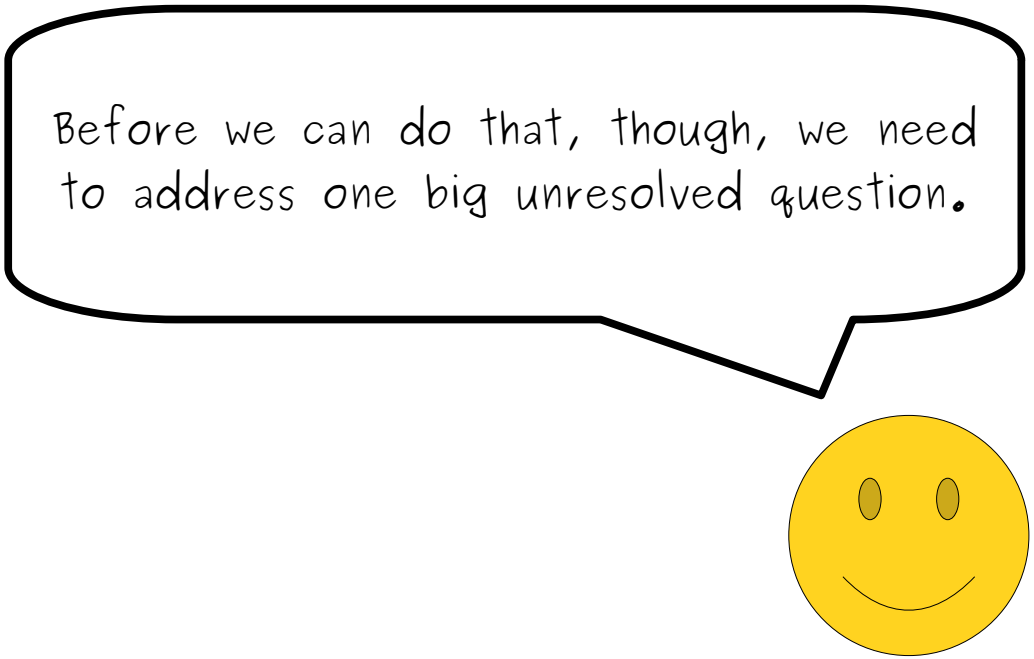

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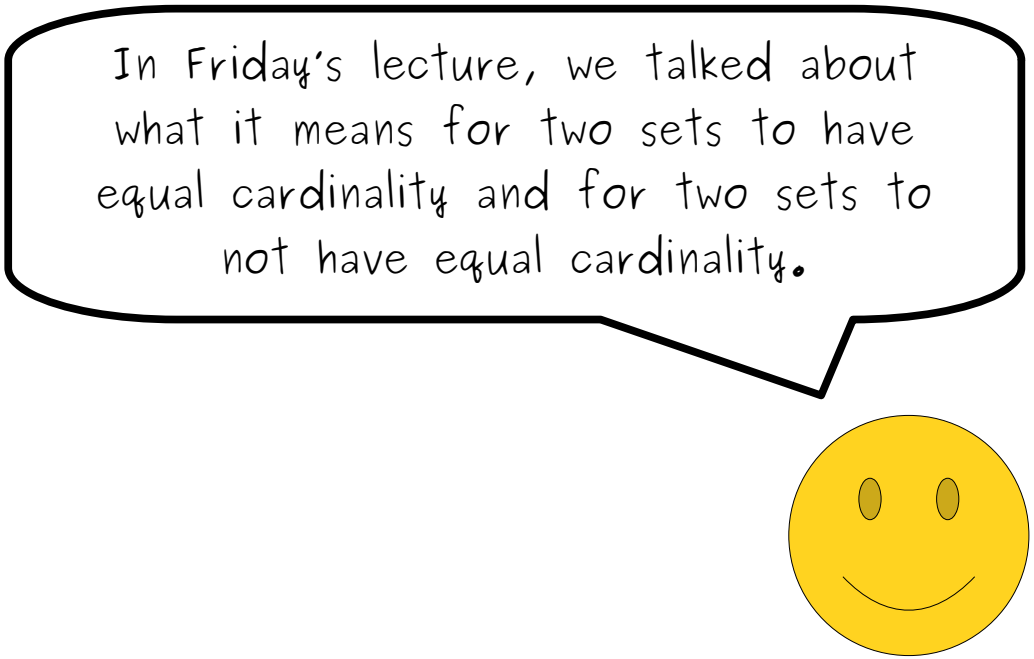

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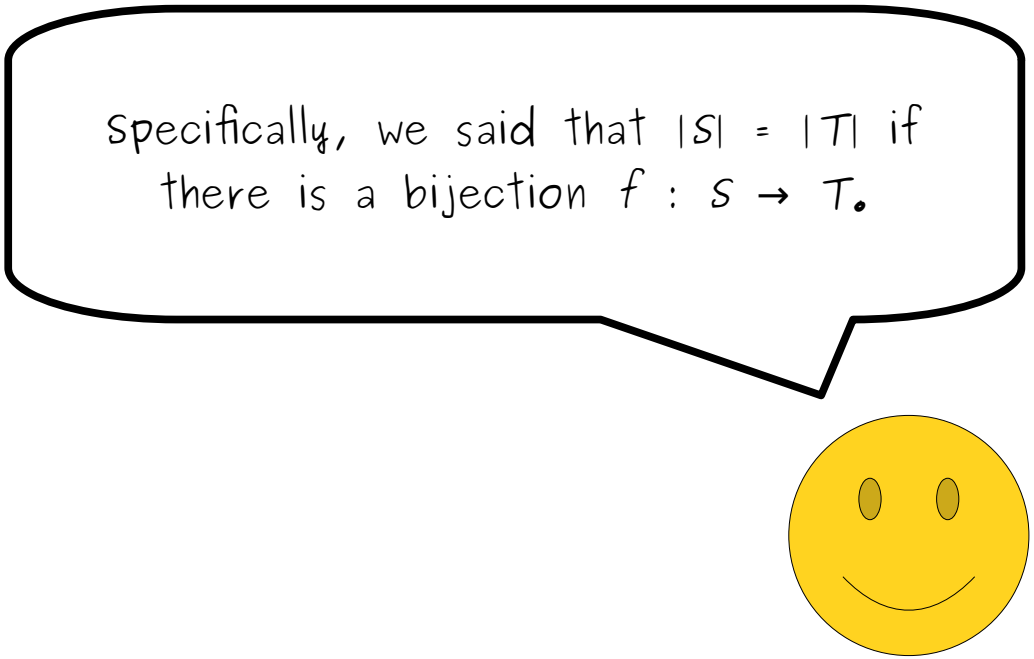
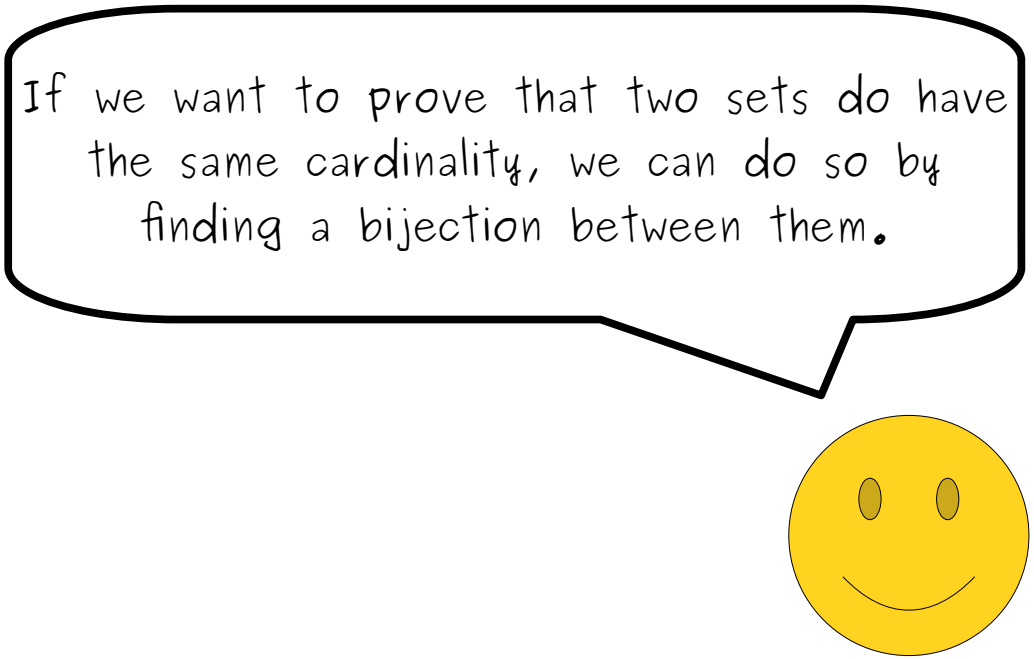


If S is a set, then $|S| < |\wp(S)|$

As a result, $|S| \neq |T|$ if there are no bijections at from S to T.

If S is a set, then $|S| < |\wp(S)|$



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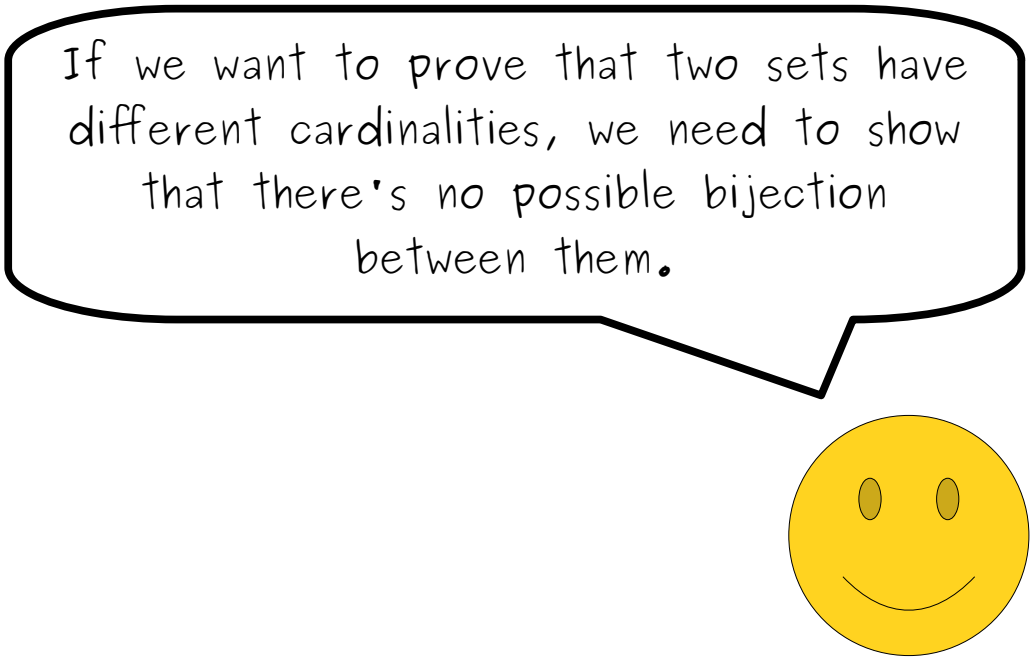

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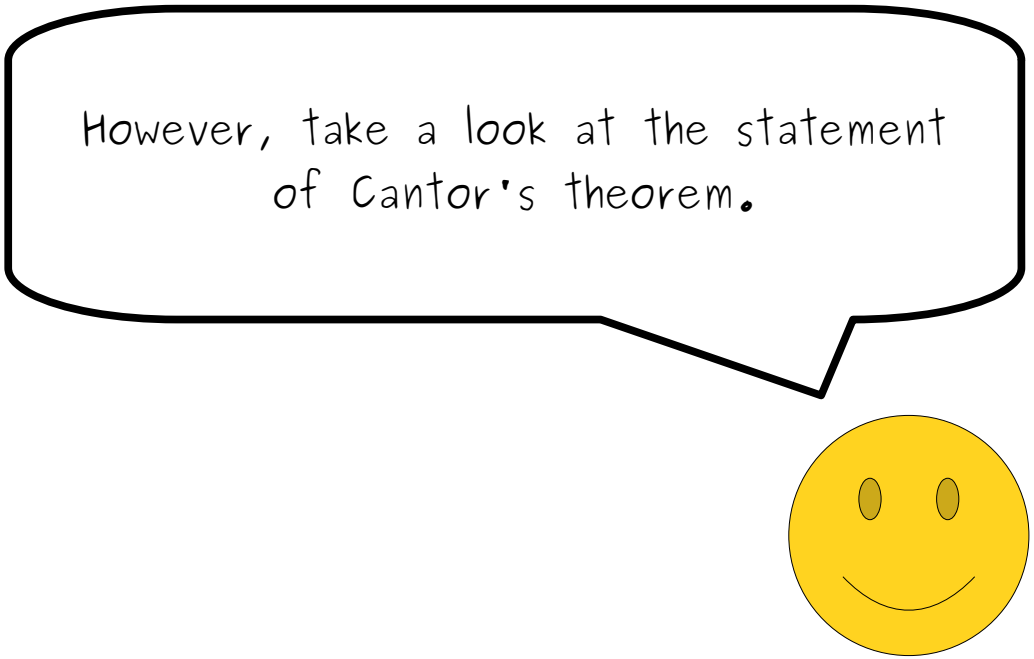

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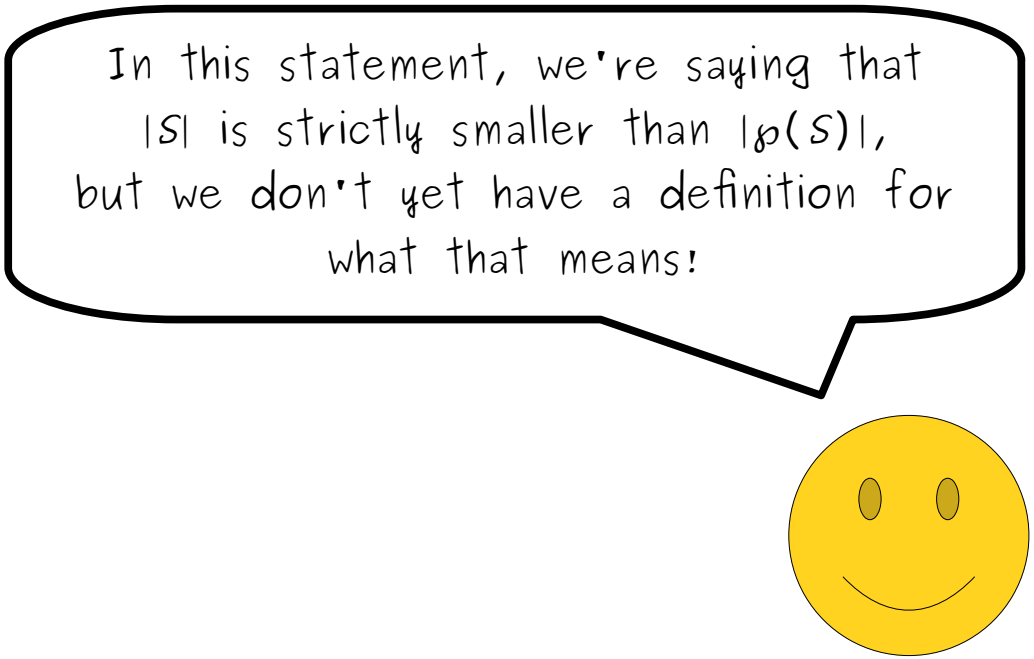

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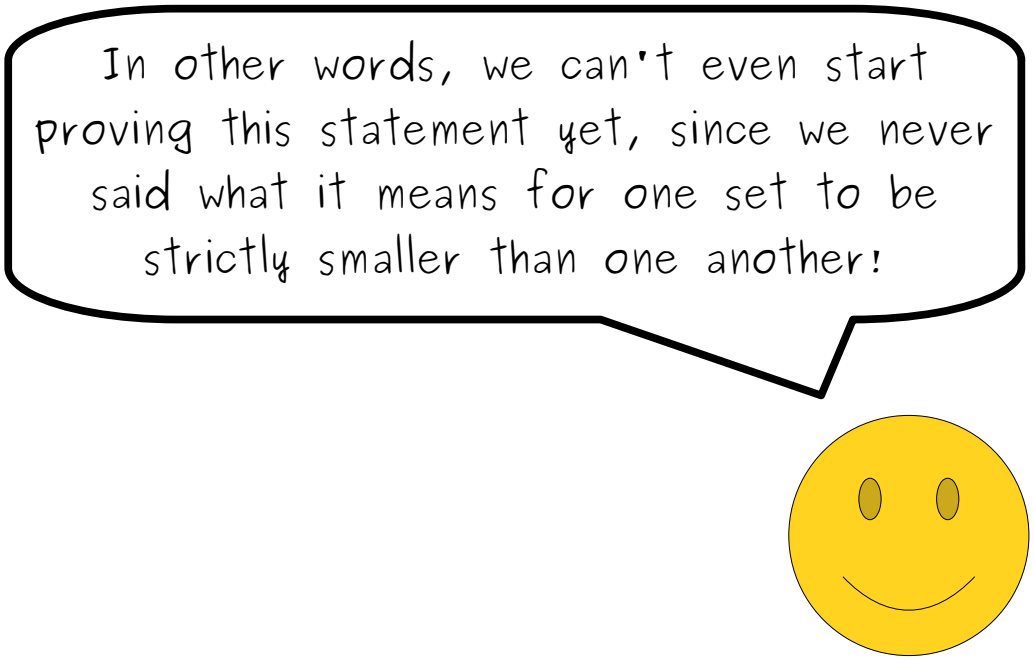

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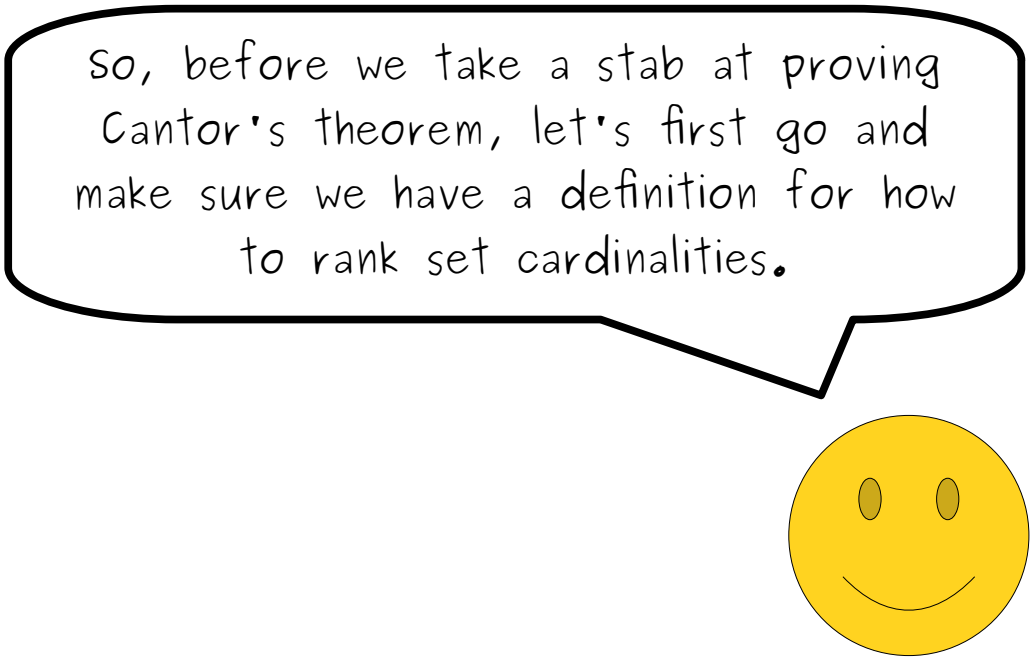
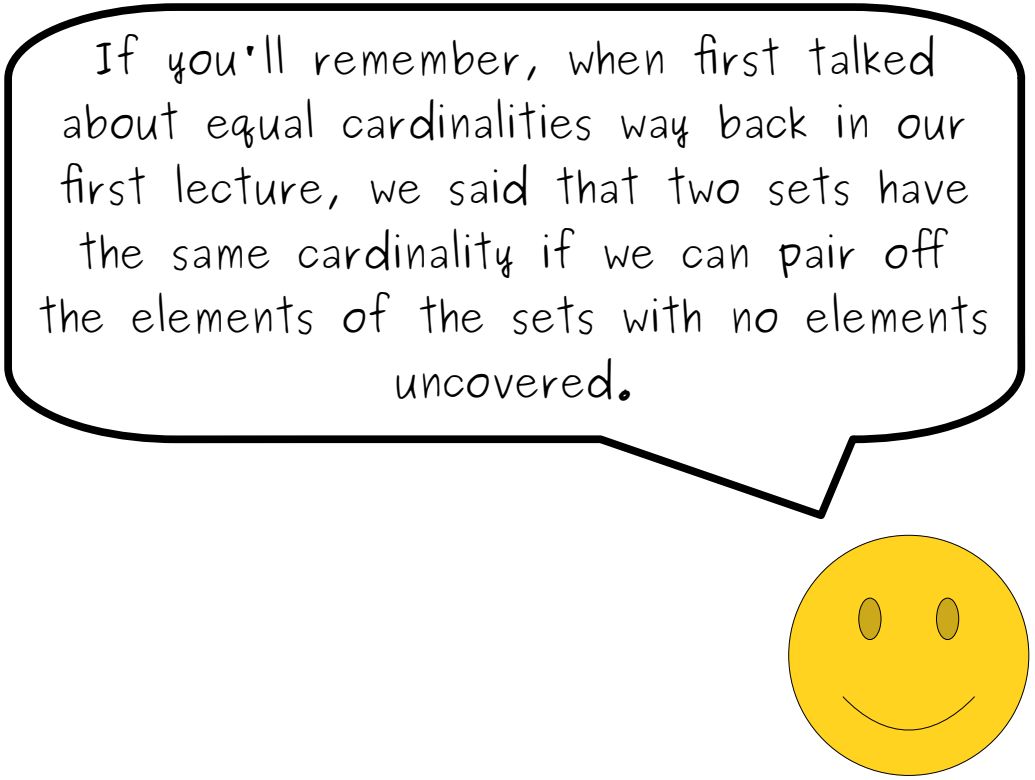
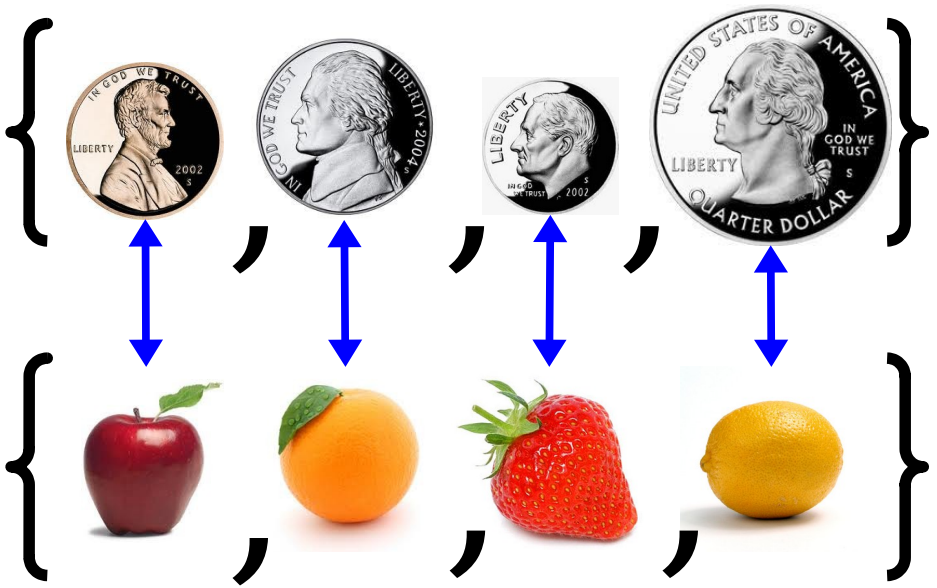
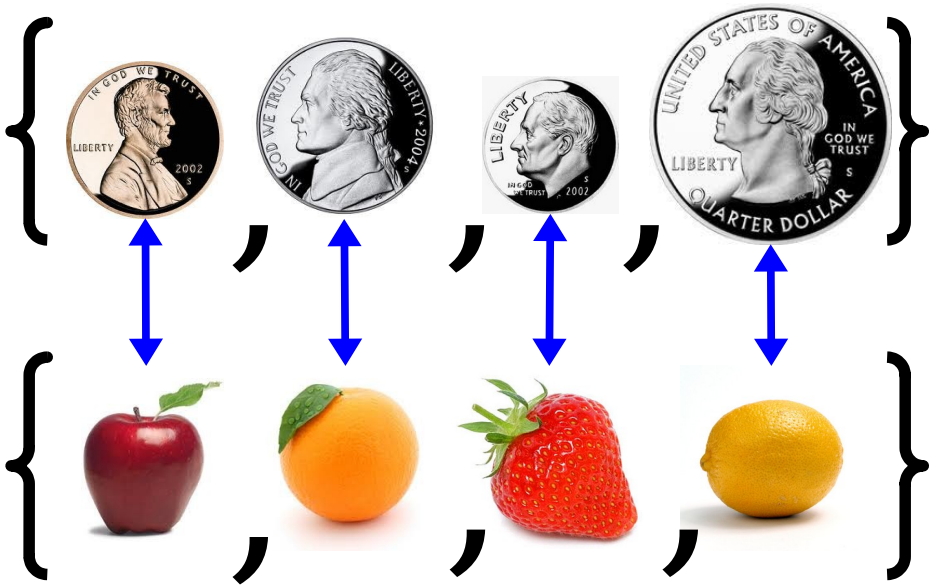


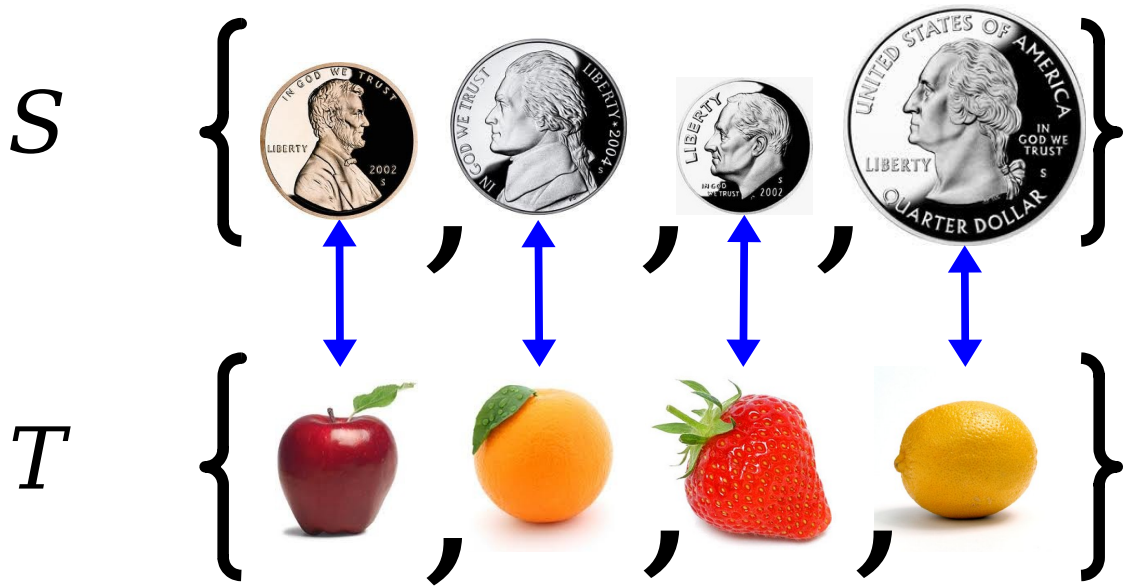

If you'll remember, when first talked about equal cardinalities way back in our first lecture, we said that two sets have the same cardinality if we can pair off the elements of the sets with no elements uncovered.



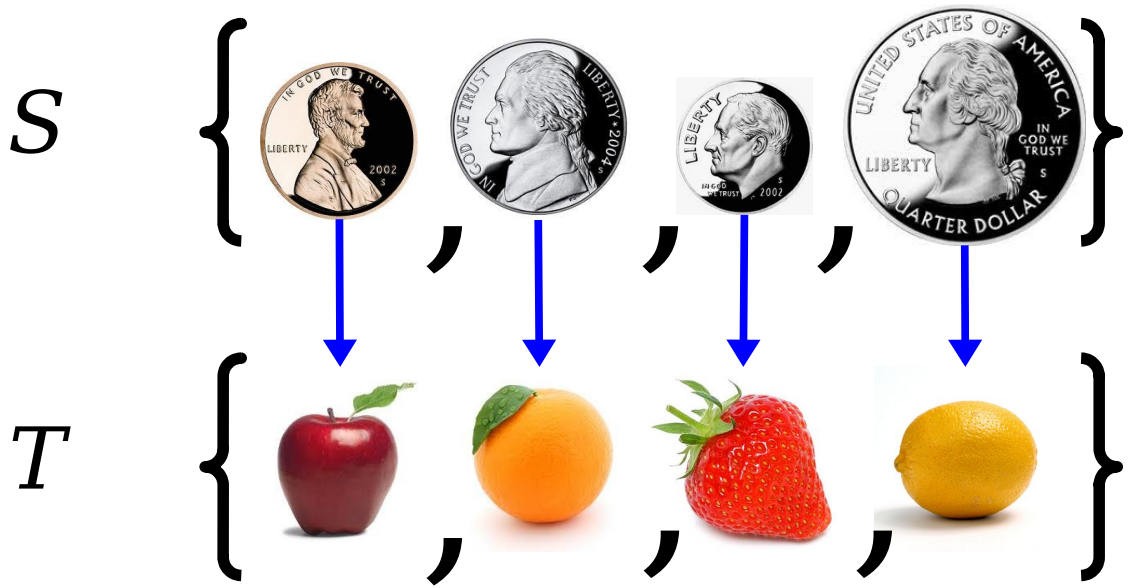
That's why these two sets have the same cardinality.



Once we had the language of bijections, we were able to give a more formal definition of what "pairing things off" means.

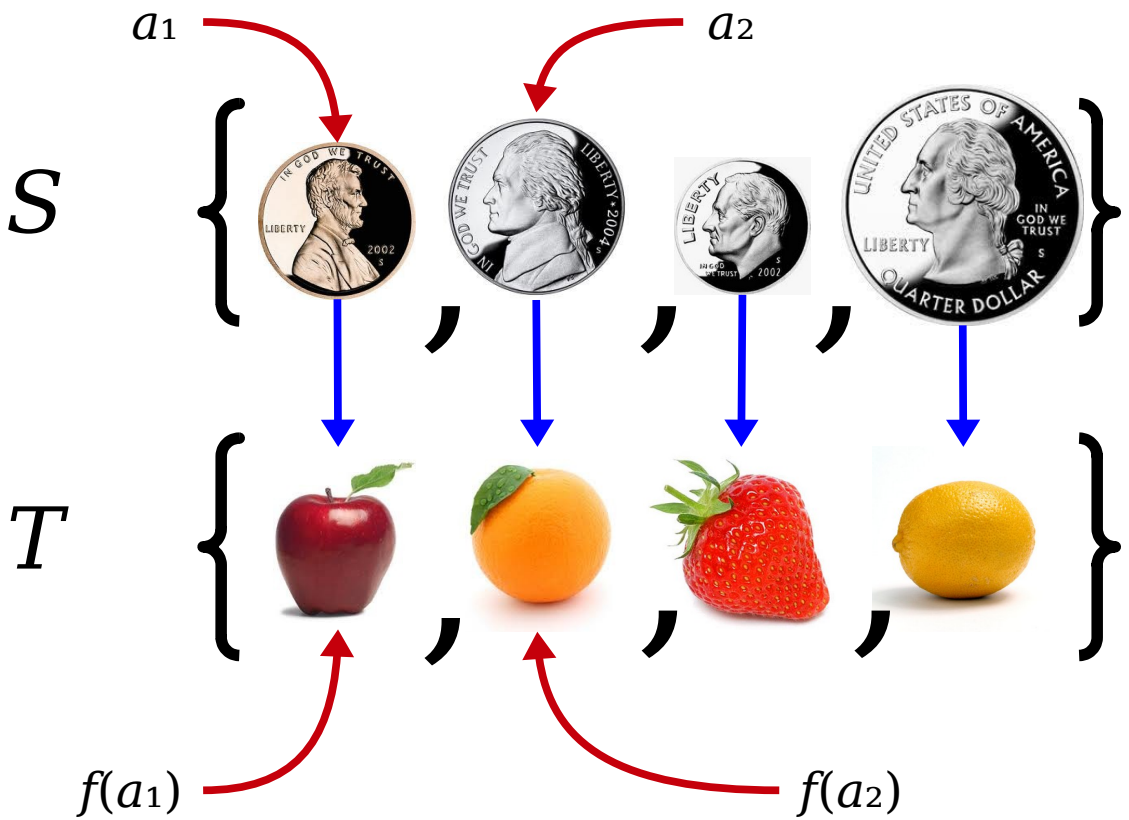


Specifically, if we label these two sets S and T ...



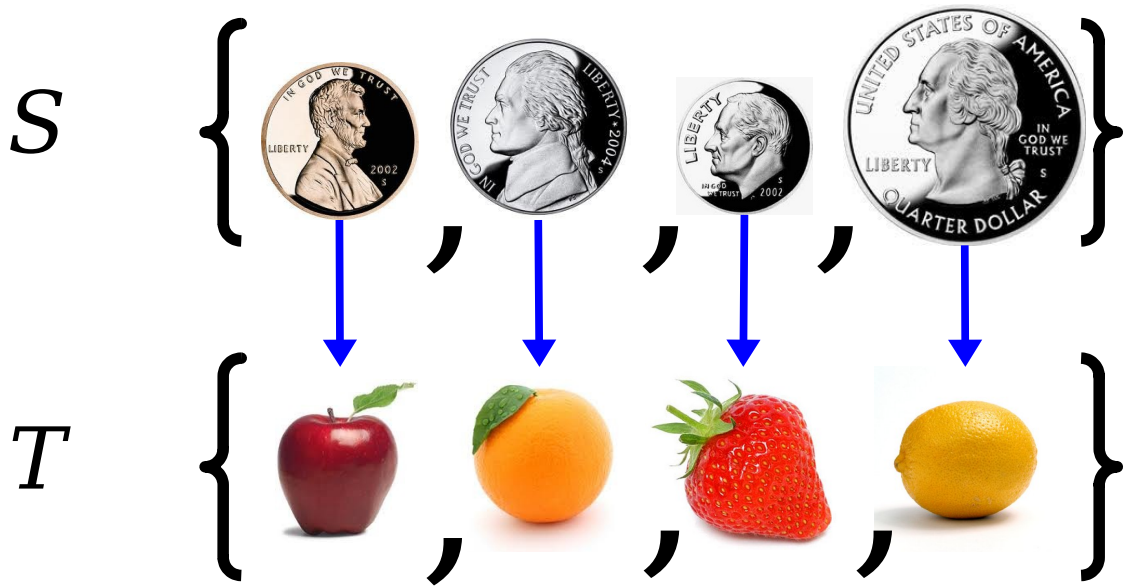
$f : S \to T$

We can think of the pairing as a function from S to T.



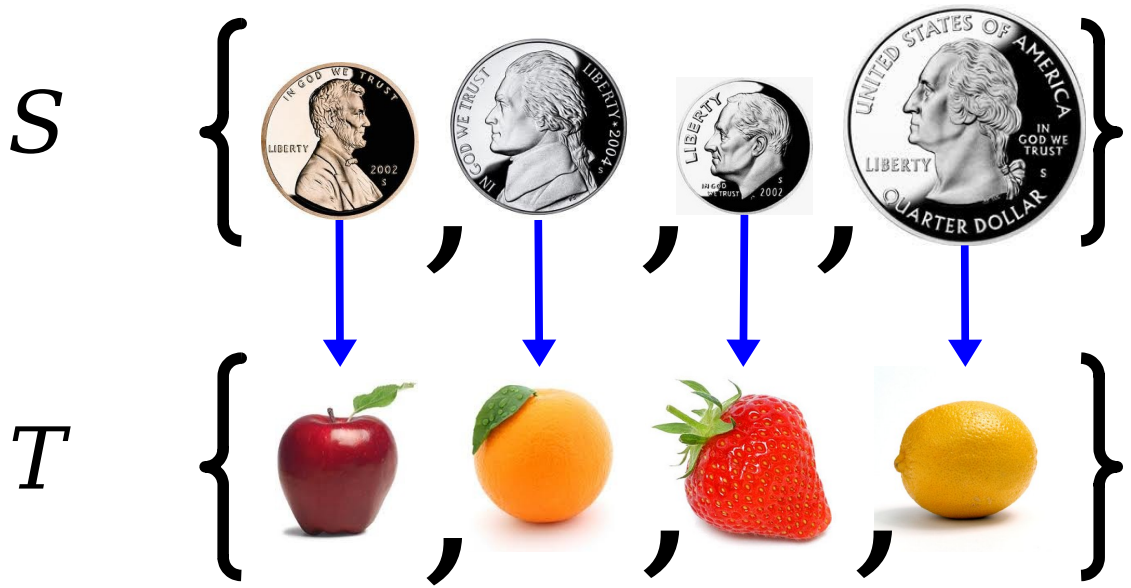
$f : S \to T$

This function will be injective – any two different inputs produce different outputs.



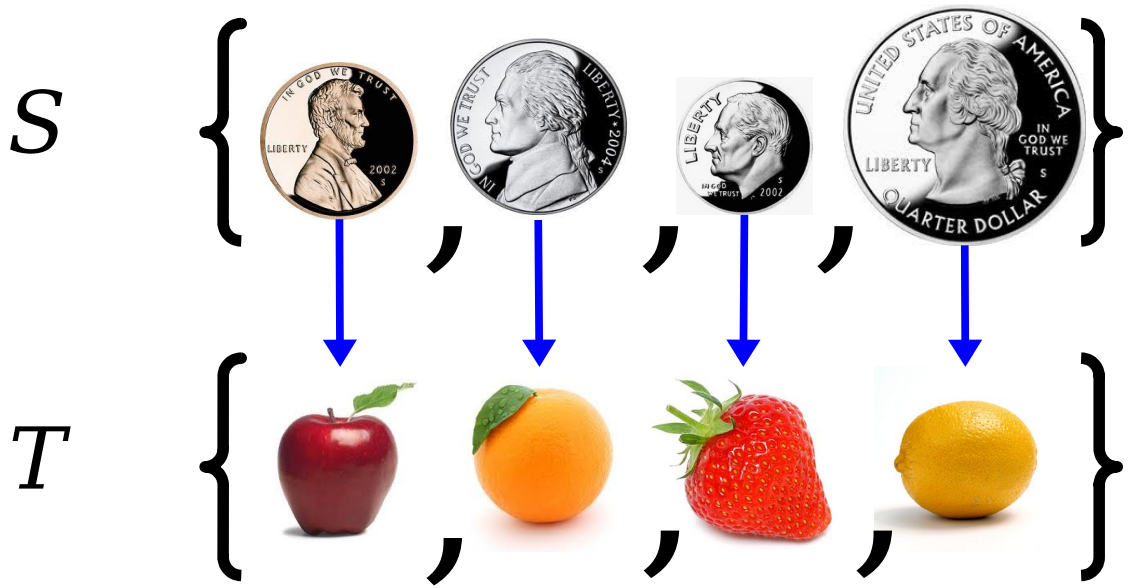
$f : S \to T$

It's also surjective, since any element of the codomain (here, T) will be paired with something in S.



$f : S \rightarrow T$

That's where we get the bijection-based definition of equal set cardinality from.

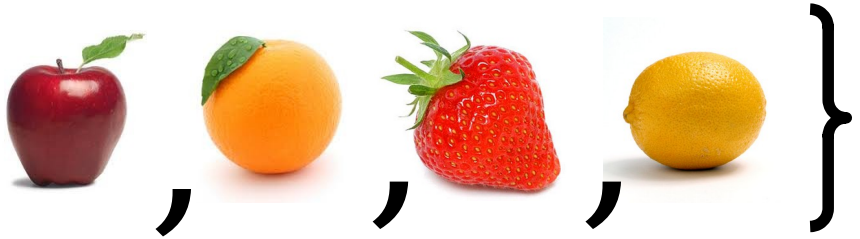


$f : S \to T$

Could we develop some sort of analogous definition for what it means for one set to be "at least as big" as another?

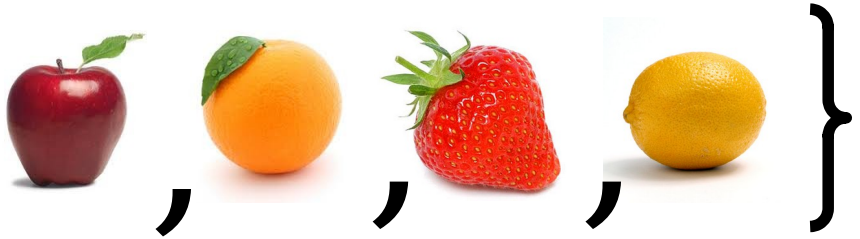
Let's begin by picking two new sets...

S

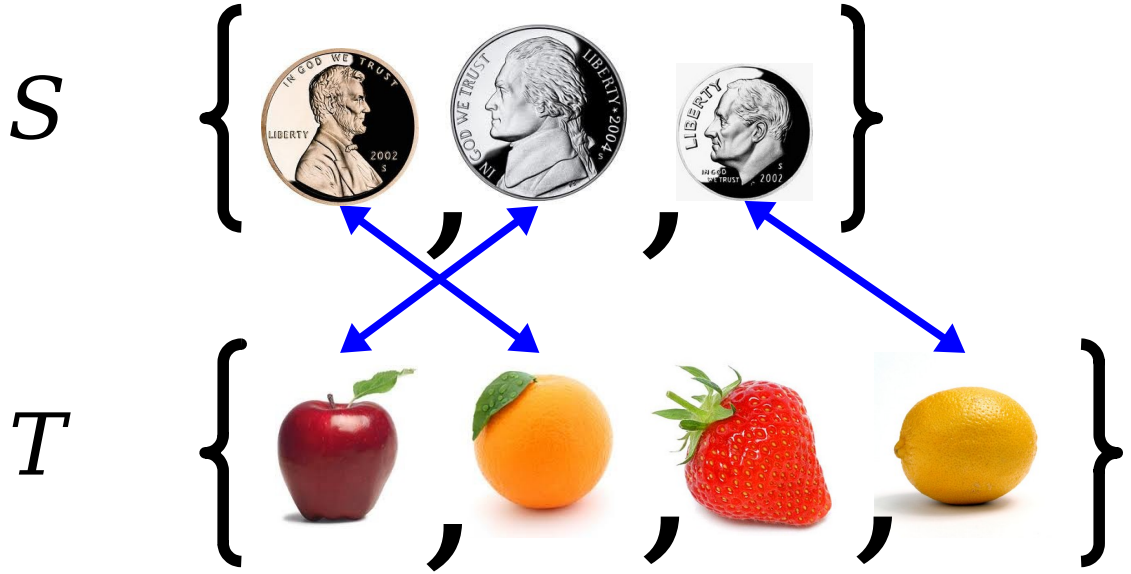
T 

… say, these two sets here.

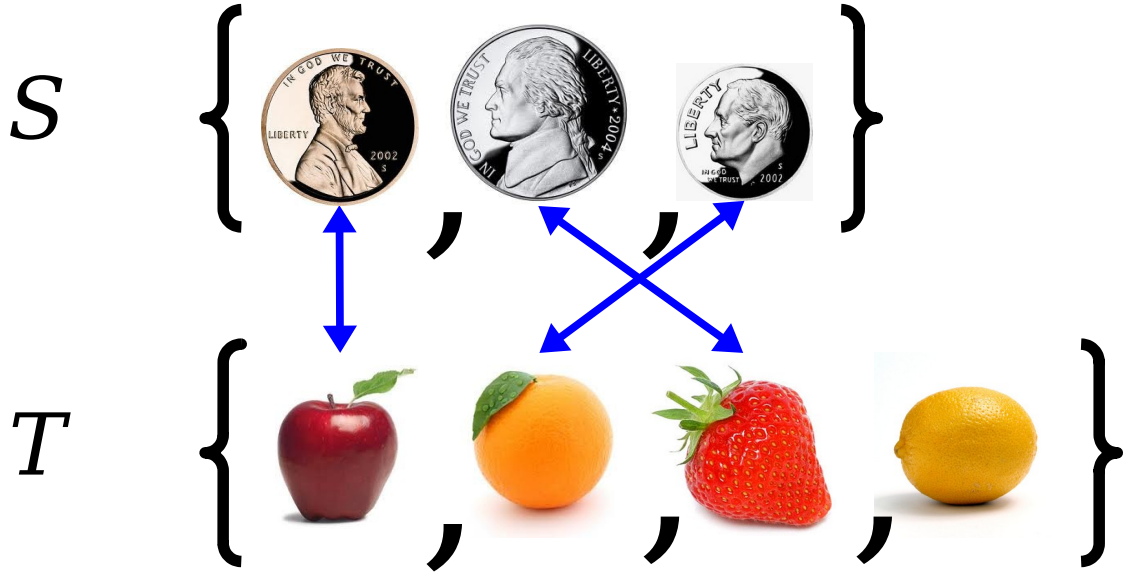
S

T 

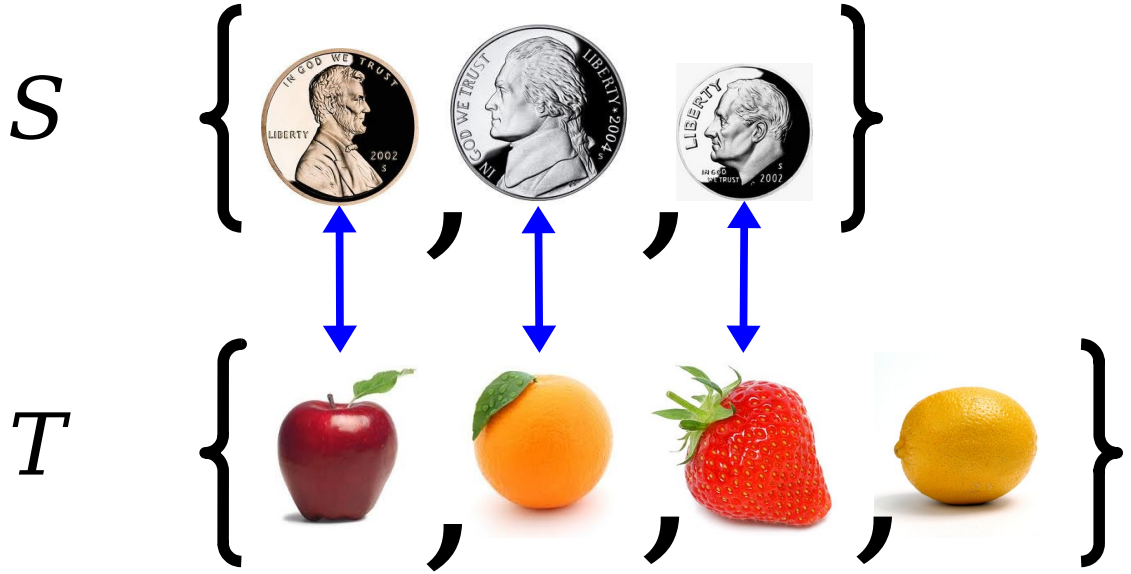
If we try pairing off the elements of S and the elements of T...



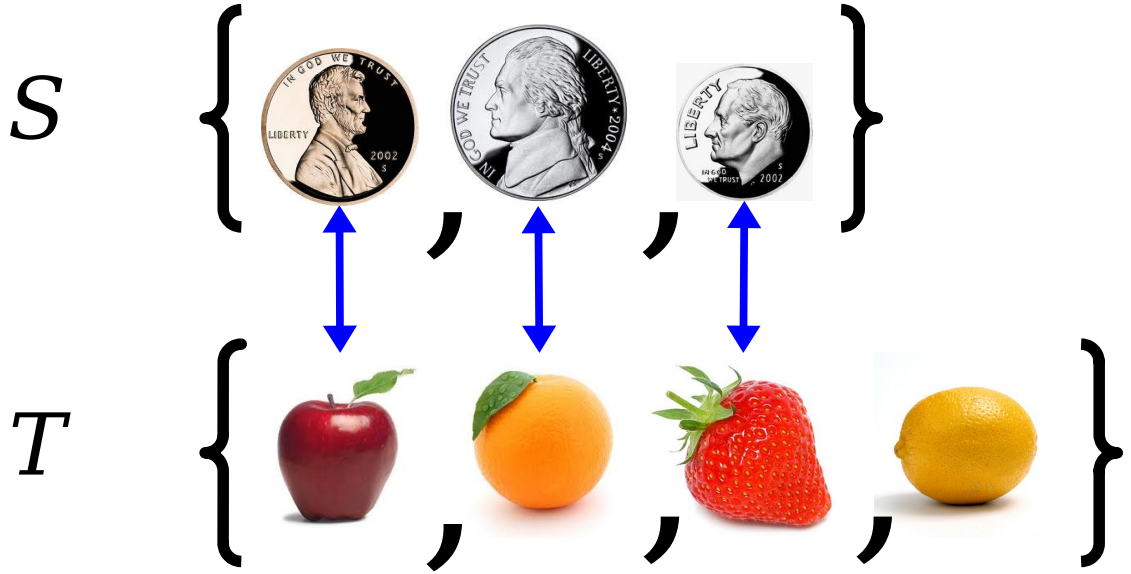
…we can see that there's some way to do it that uses up everything from S.



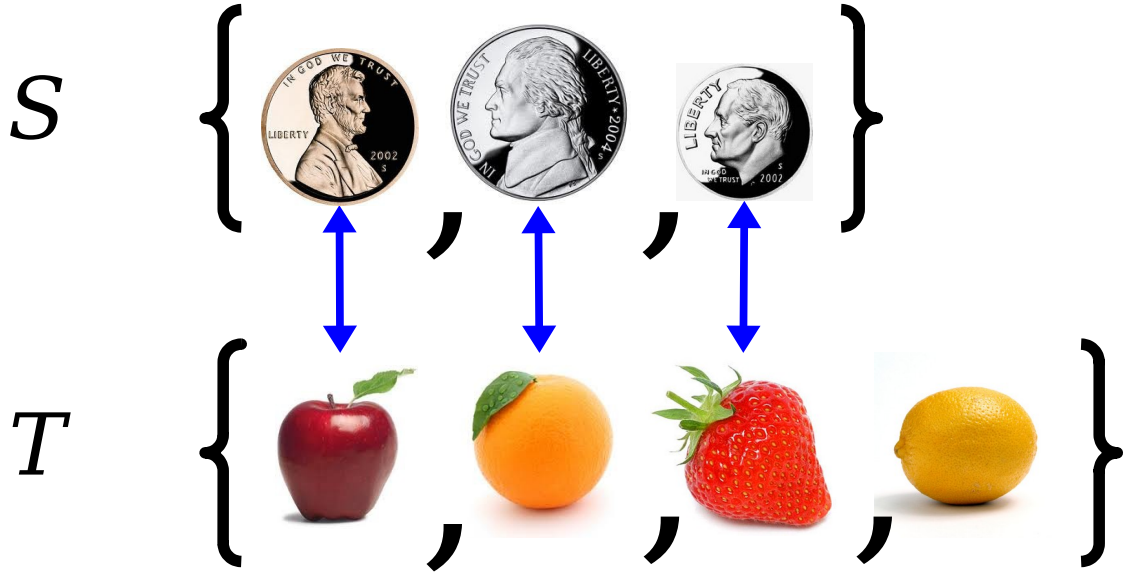
There's actually a bunch of ways to do this. Here's a different one...



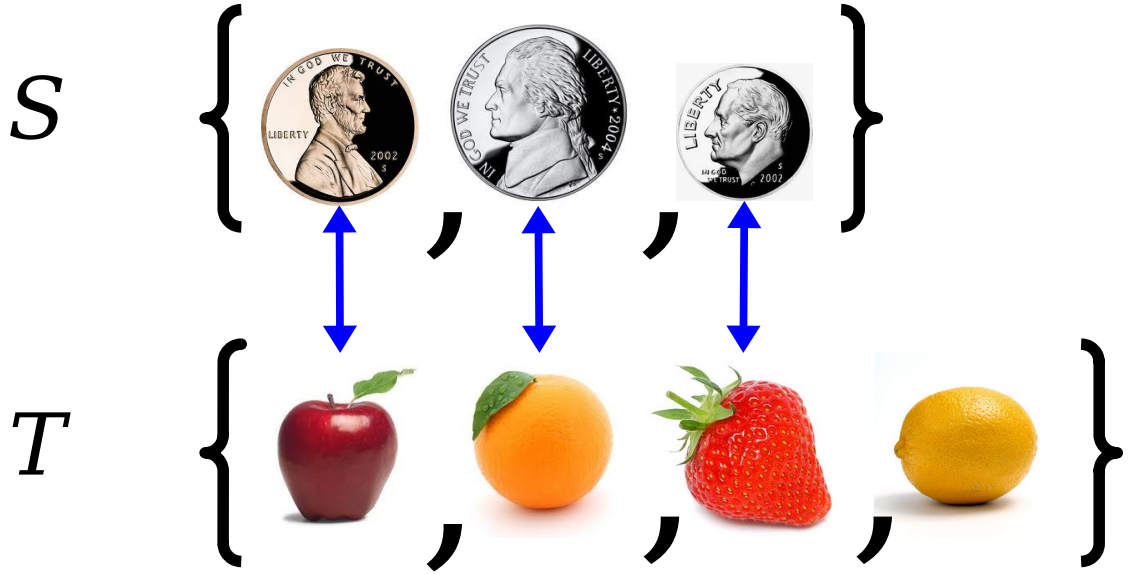
…and here's one more.



This gives us a nice intuition for what it means for one set to have at least as many elements as the other.



Specifically, we'll say that T is at least as big as S if there's a way to pair the elements that uses up everything in S, even if it doesn't use up everything in T.



Before moving on, let's make sure that this definition works in some other cases. If it doesn't, then chances are it's not a very good definition!

$\mathbb{N}$ 0 1 2 3 ...

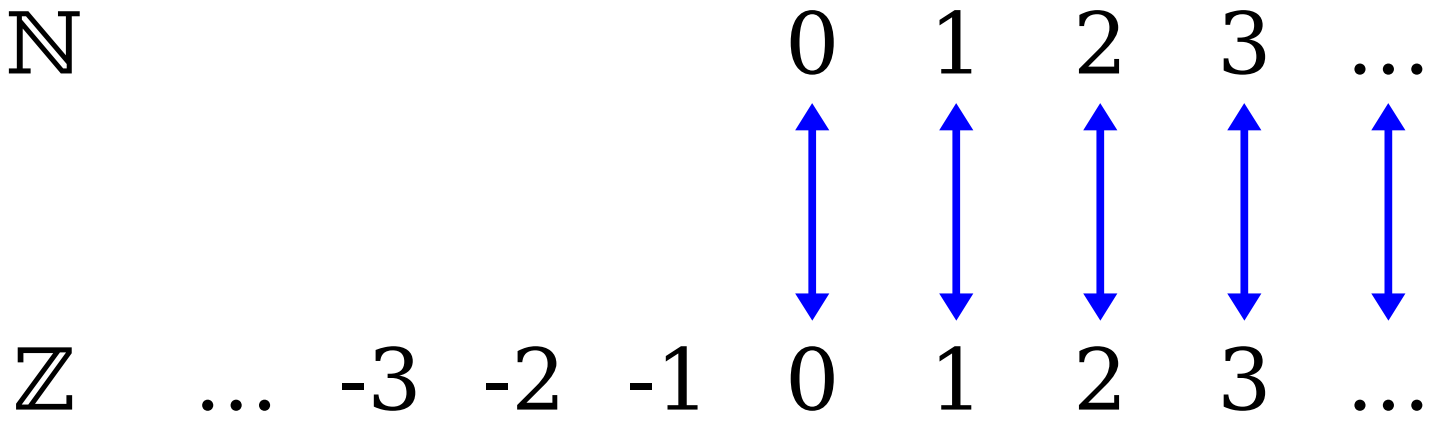
$\mathbb{Z}$ ... -3 -2 -1 0 1 2 3 ...

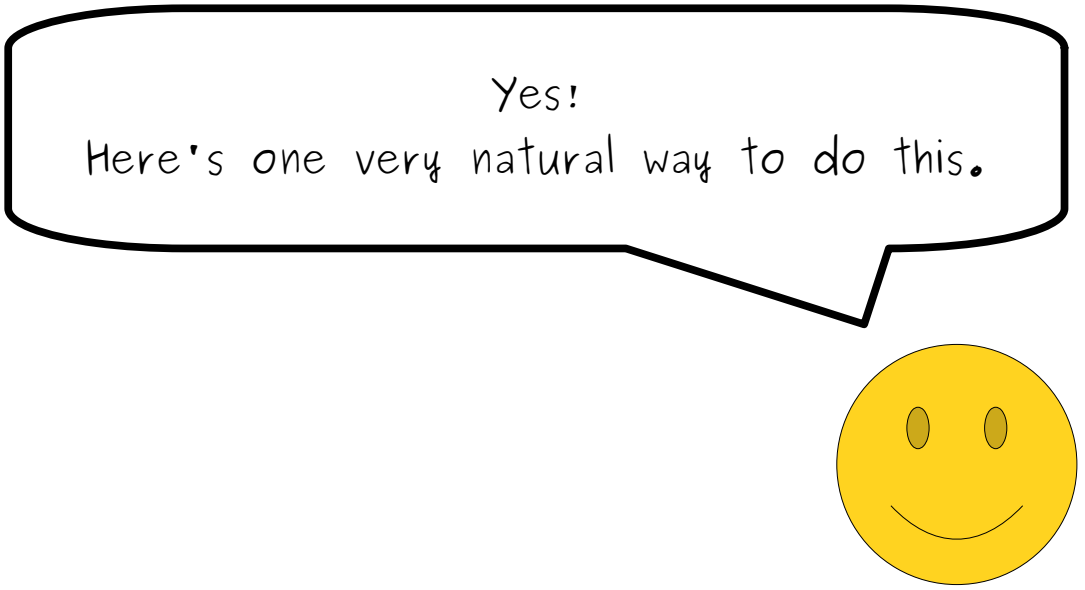
For example, let's take a look at the sets $\mathbb{N}$ and $\mathbb{Z}$.

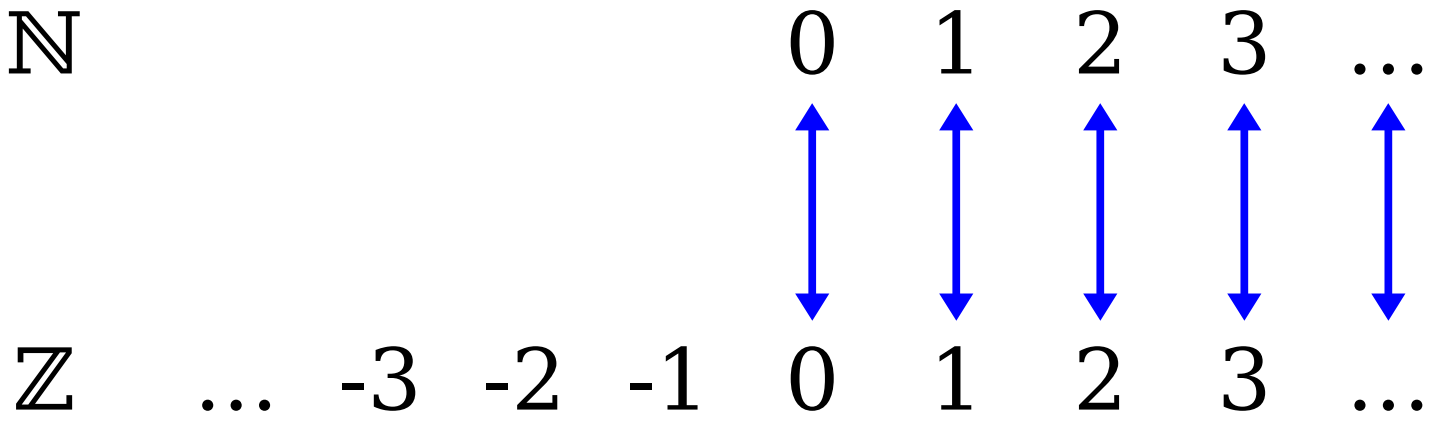
$\mathbb{N}$ 0 1 2 3 ...

$\mathbb{Z}$ ... -3 -2 -1 0 1 2 3 ...

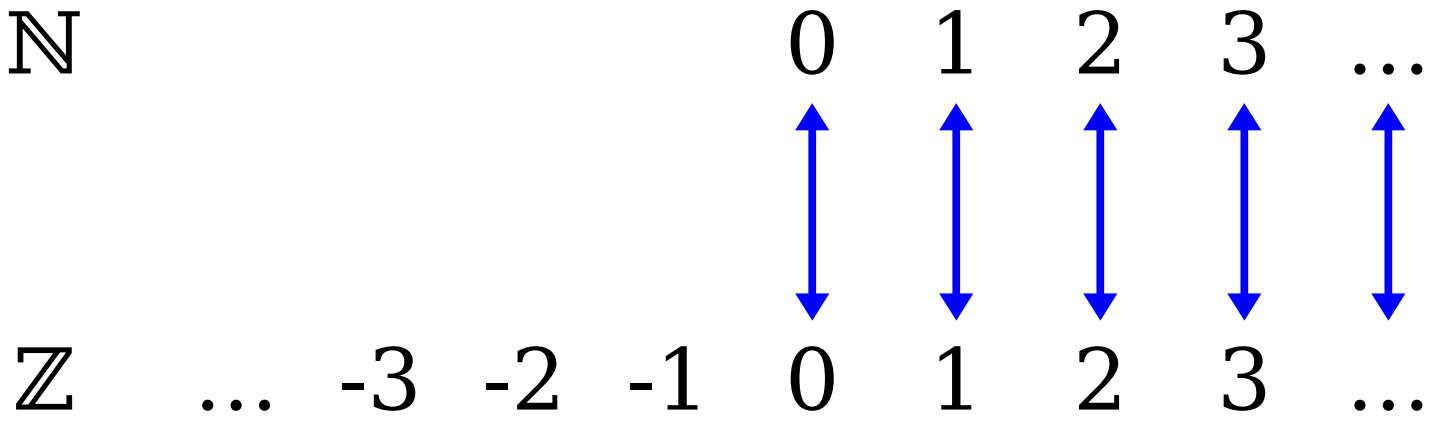
Can we pair off the naturals and the integers so that every natural number is paired off with some integer?


ℕ
0 1 2 3 ...
ℤ ... -3 -2 -1 0 1 2 3 ...


Yes!
Here's one very natural way to do this.



In lecture, we proved that $|\mathbb{N}| = |\mathbb{Z}|$. That means that we could in principle pair all the elements of both sets off, though it's a lot easier to do that if we don't need to pair off all the integers.


ℕ
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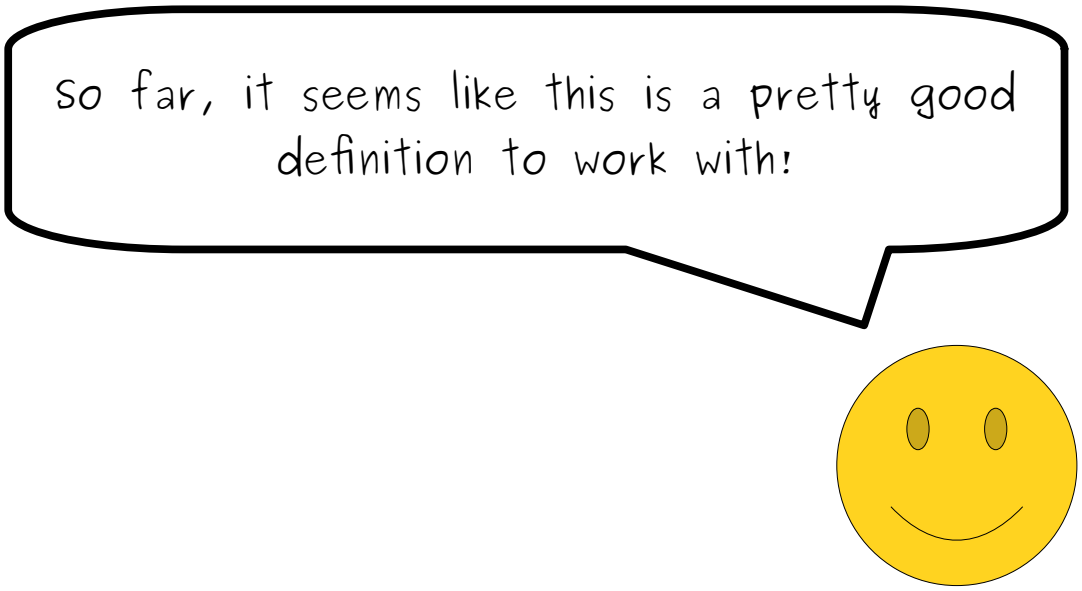
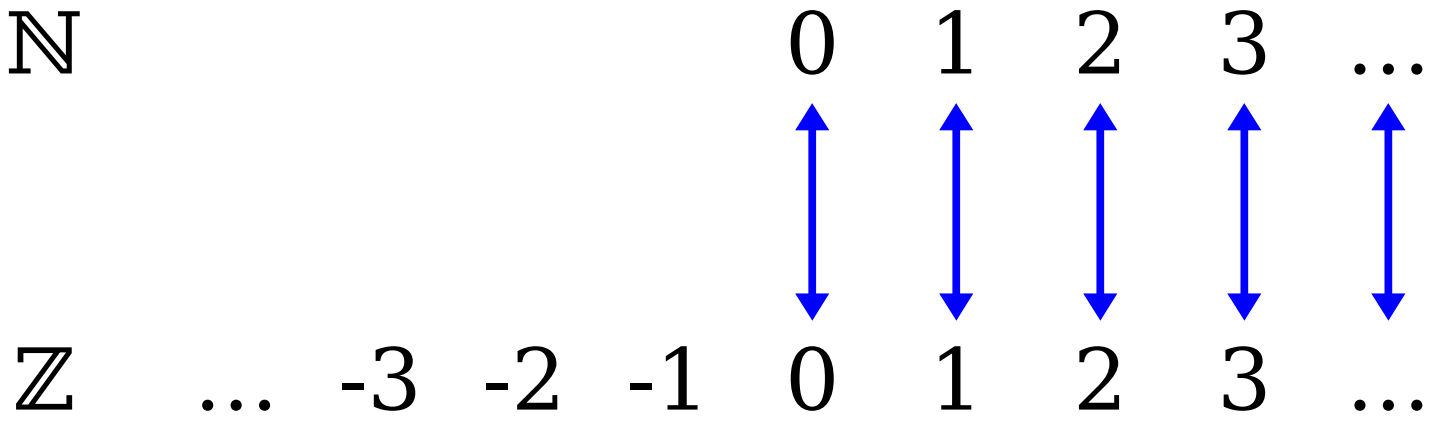
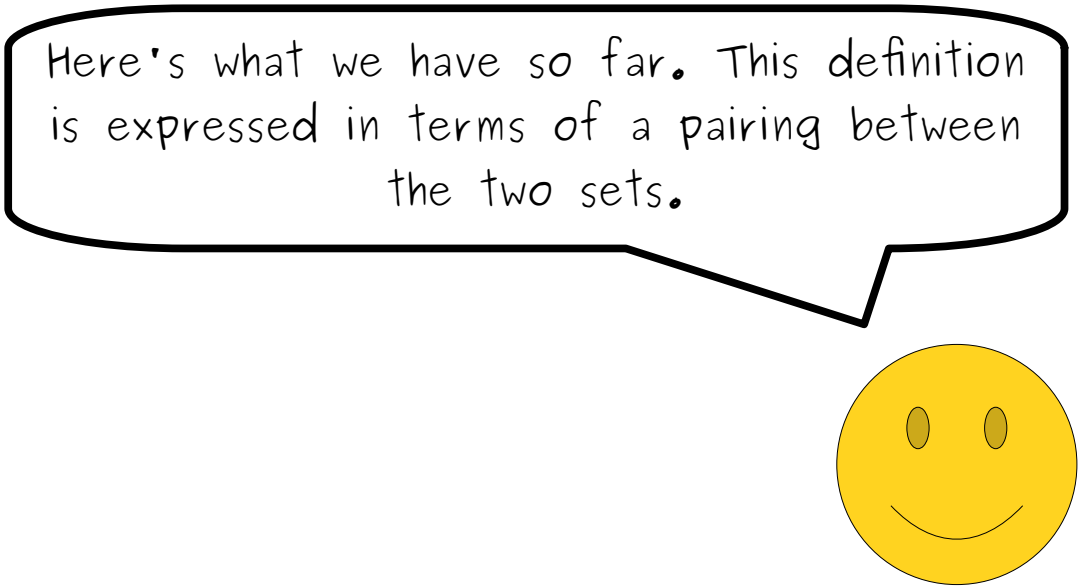
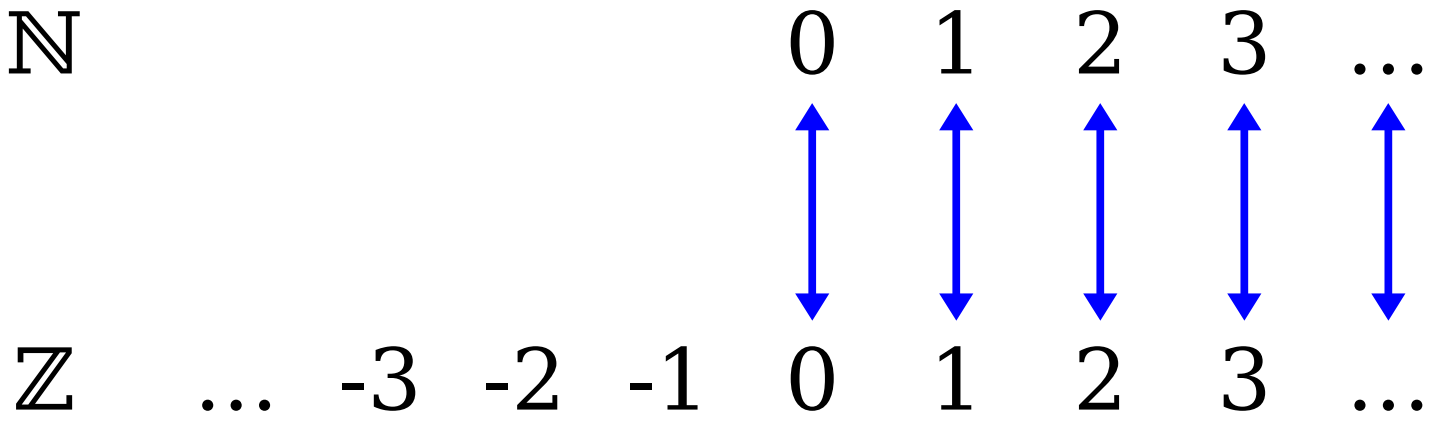

so far, it seems like this is a pretty good definition to work with!



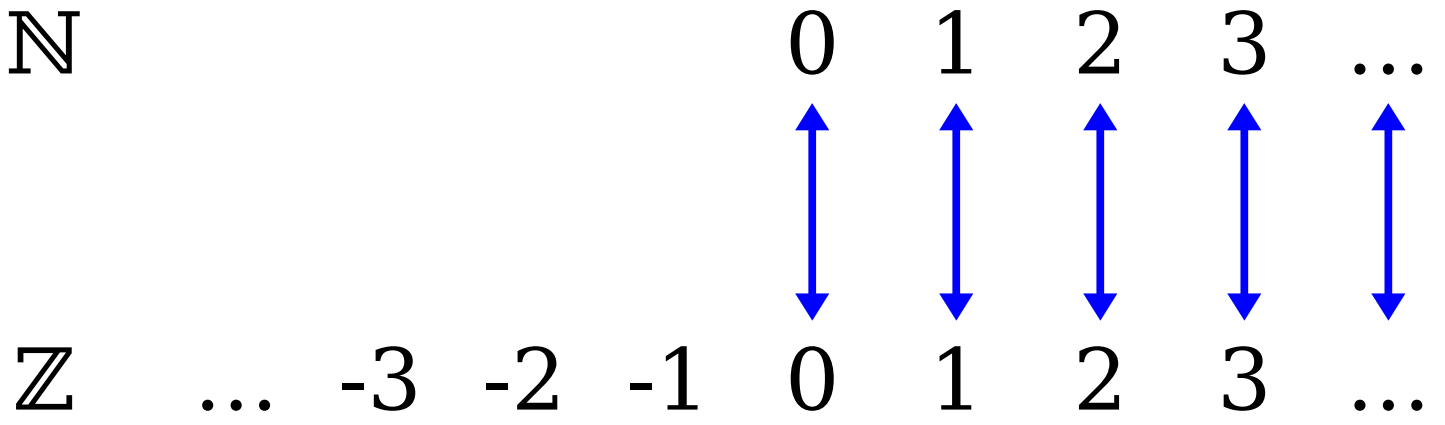
$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.





$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

When we talked about equal cardinalities, we started with a definition like this one, then revised it by talking about the pairing in terms of a certain type of function between the two sets.



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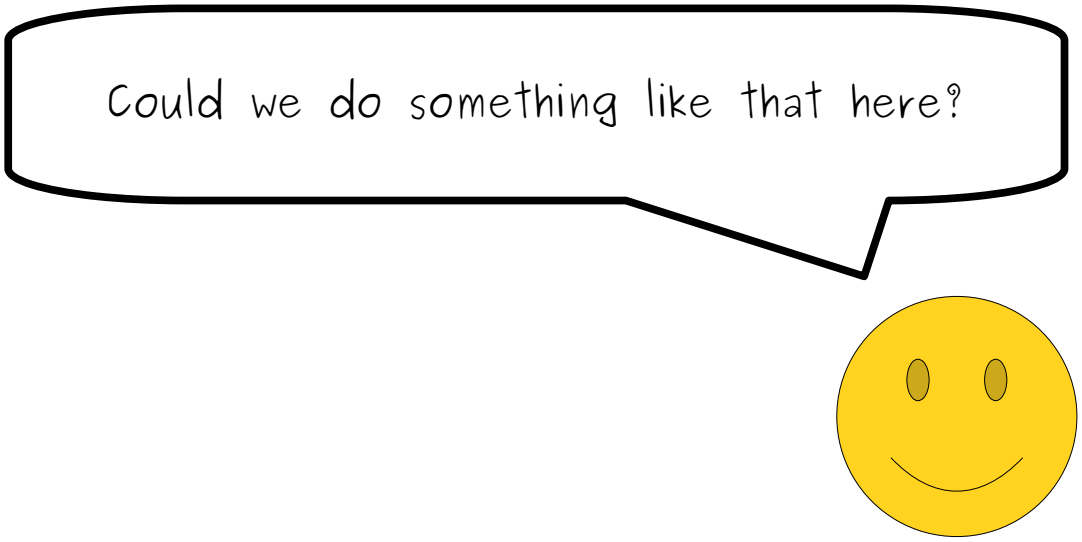

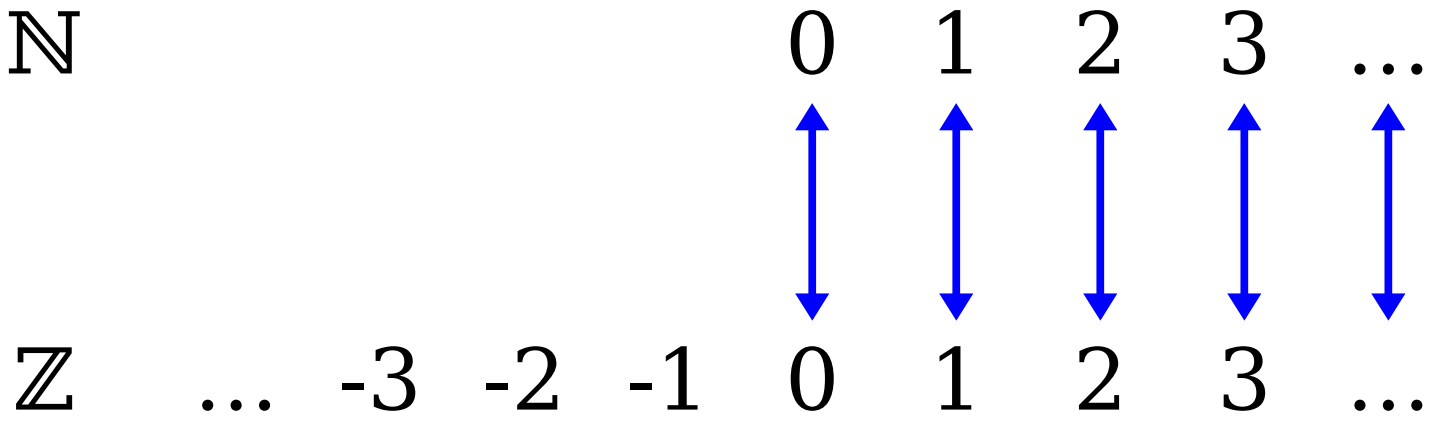

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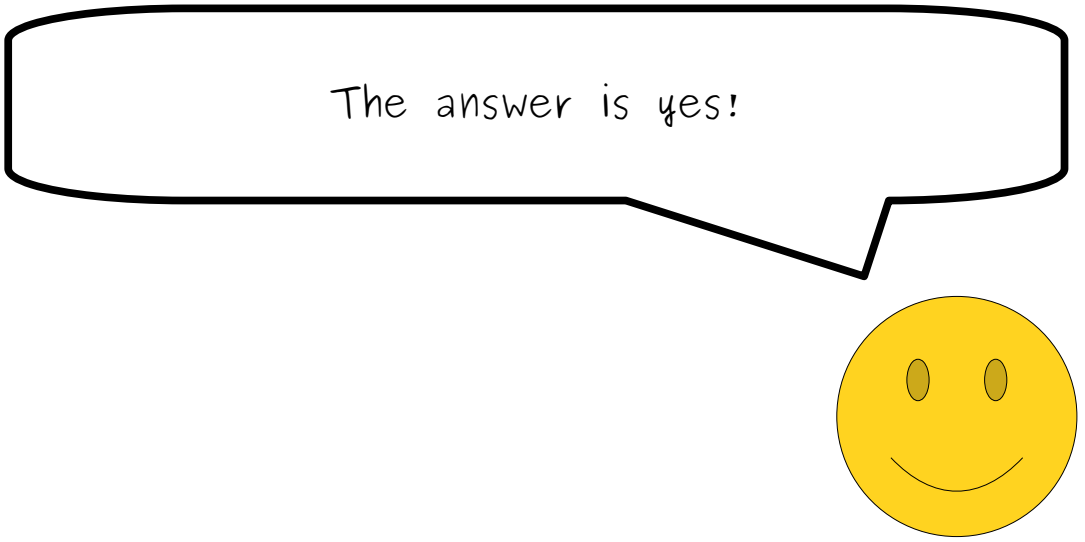

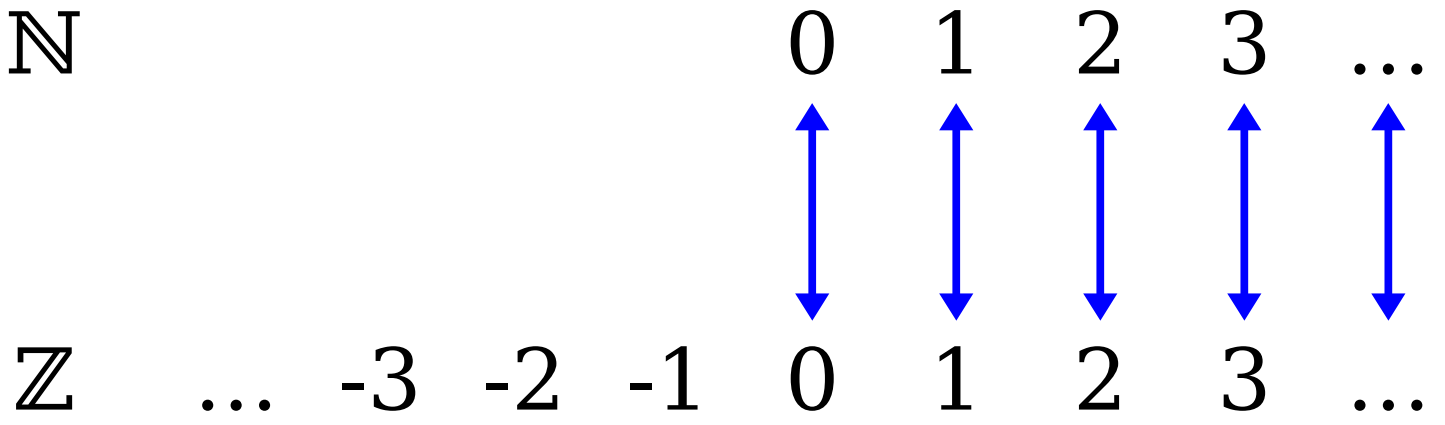

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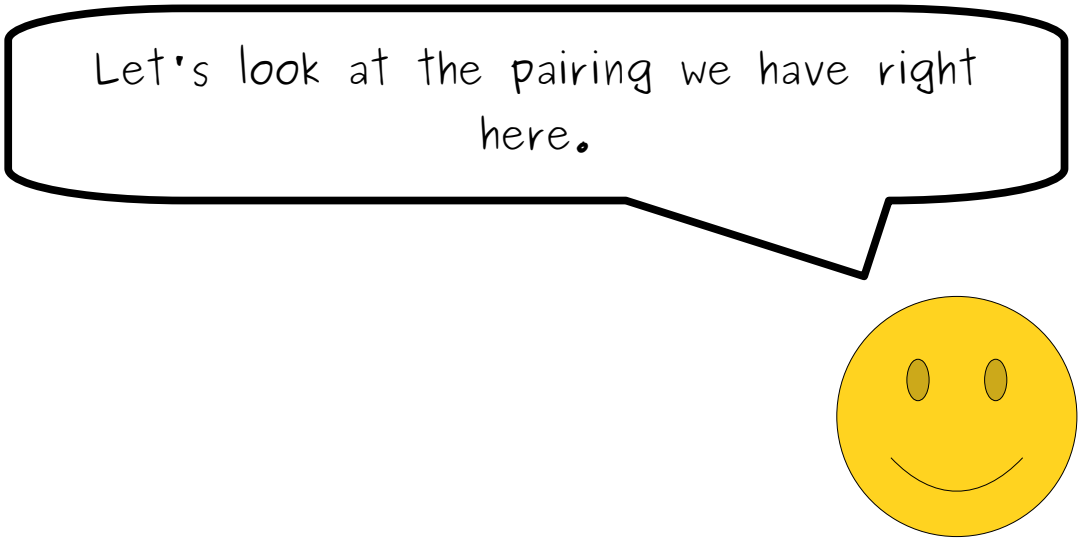
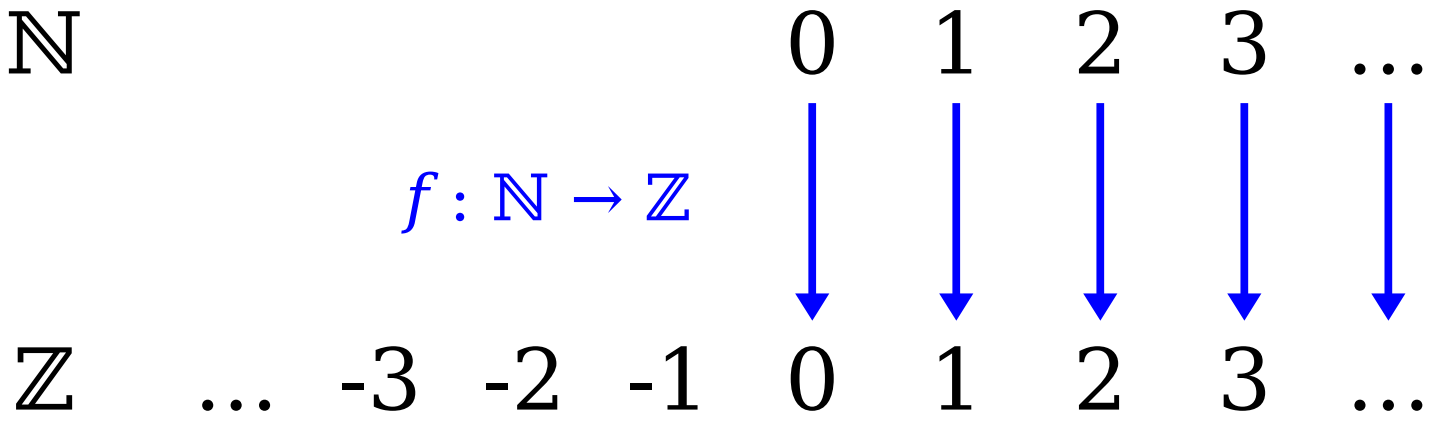
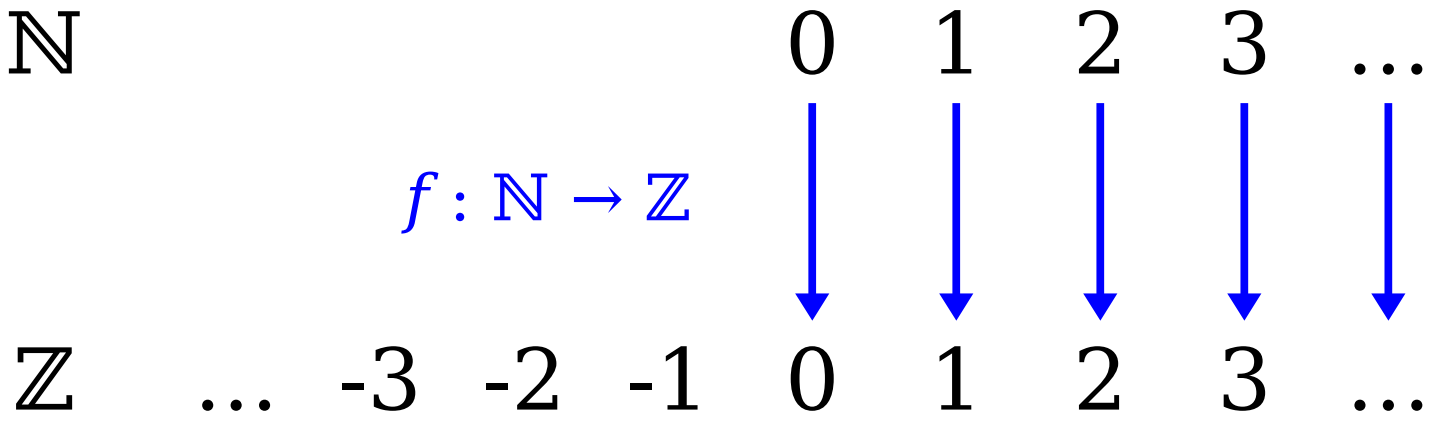




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

Let's imagine it as a function from ℕ to ℤ.



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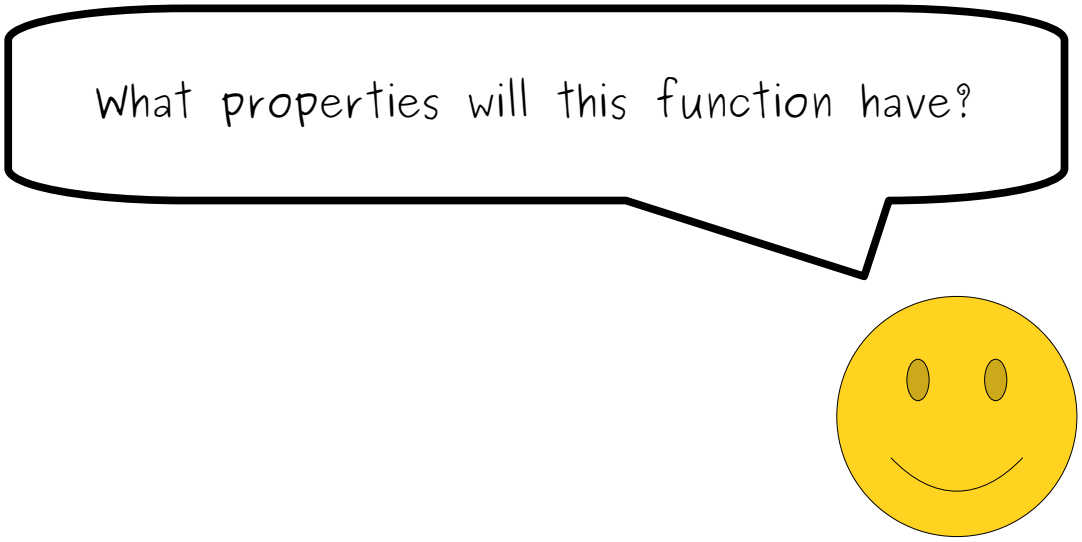
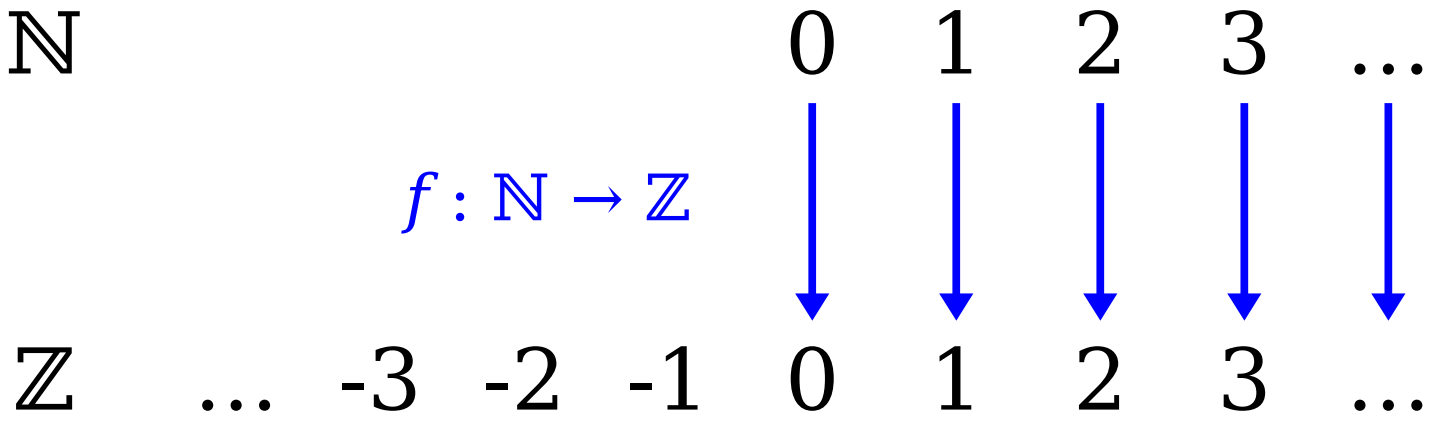
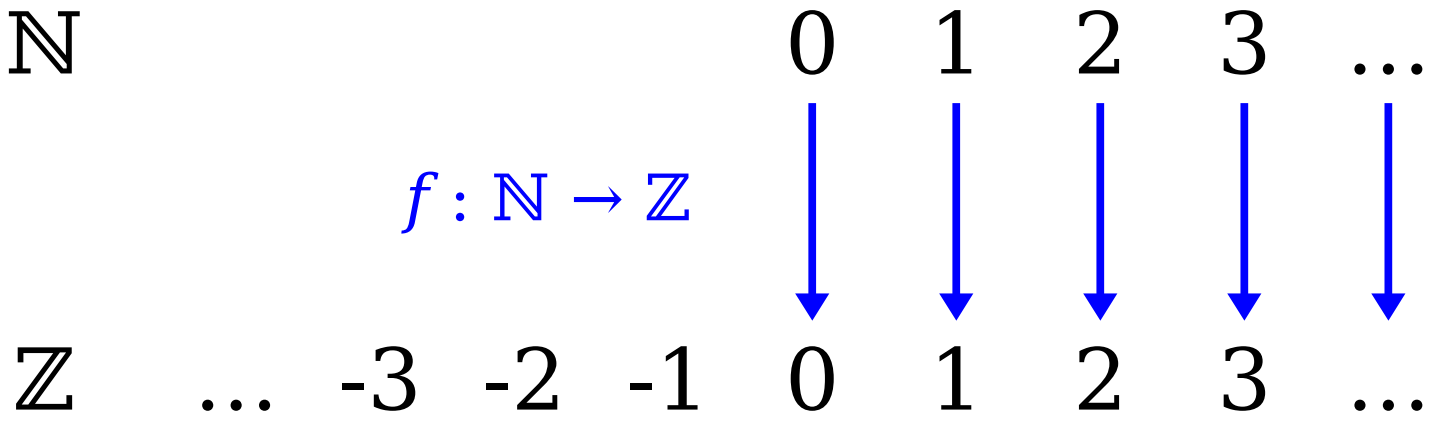




$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

Well, since it's pairing off elements of ℕ with elements of ℤ, we know that different elements of ℕ map to different elements of ℤ.



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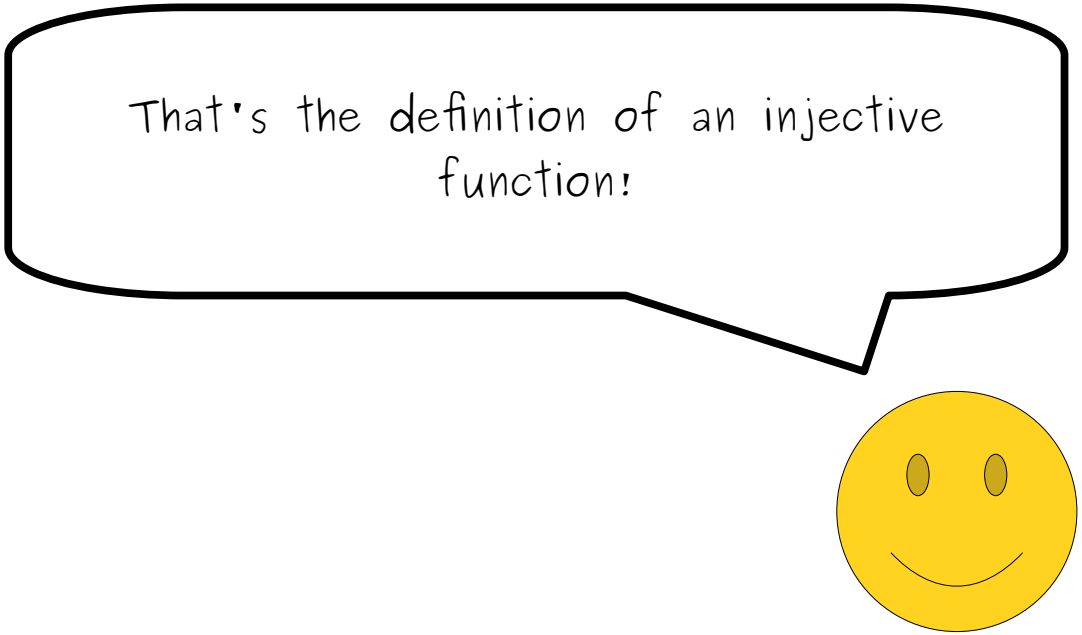
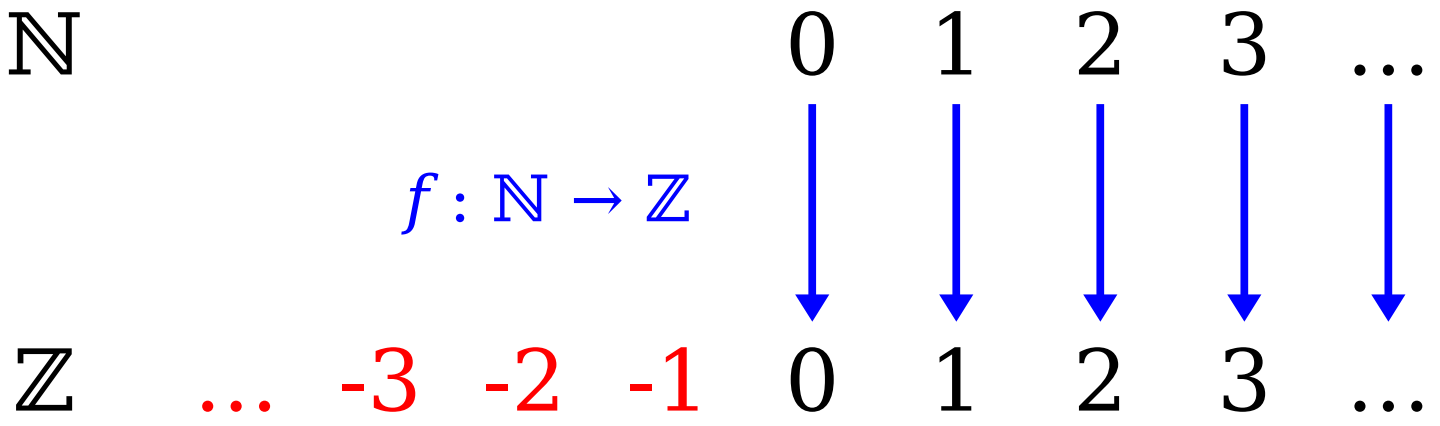
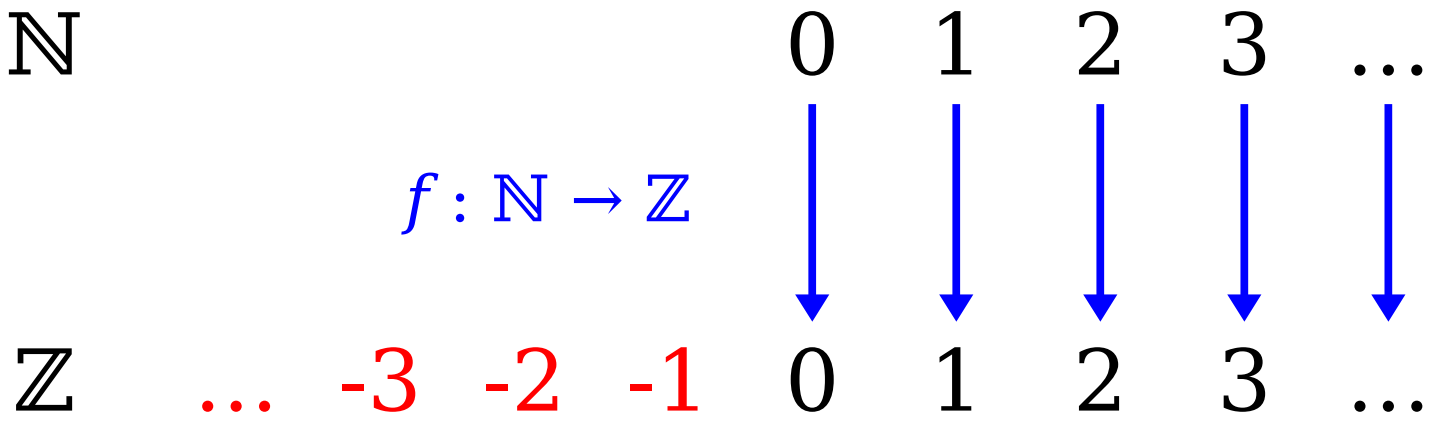




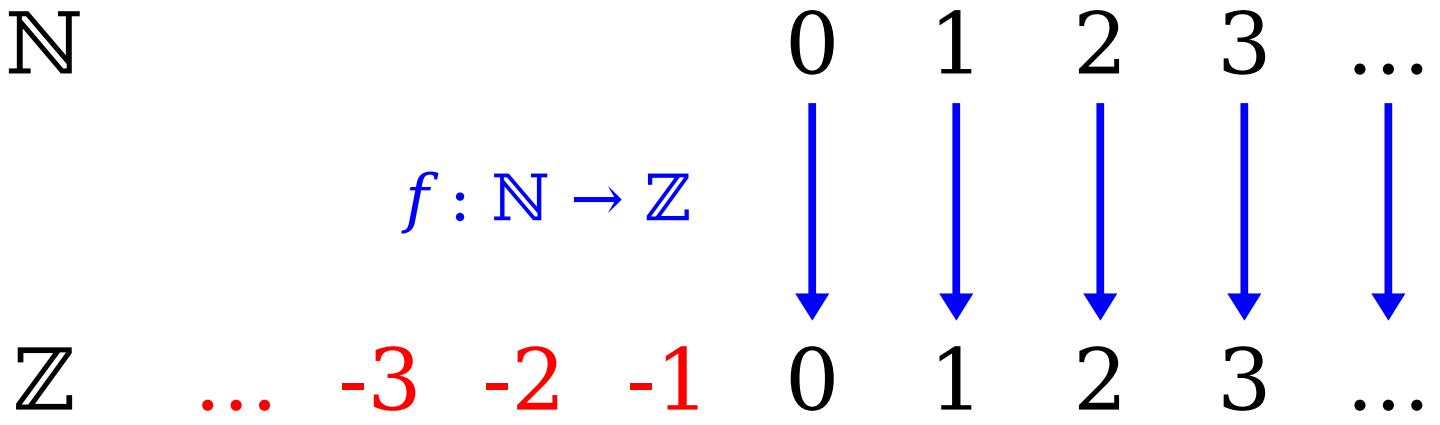
$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

However, this particular function is not surjective, since it leaves a lot of elements of ℤ uncovered.



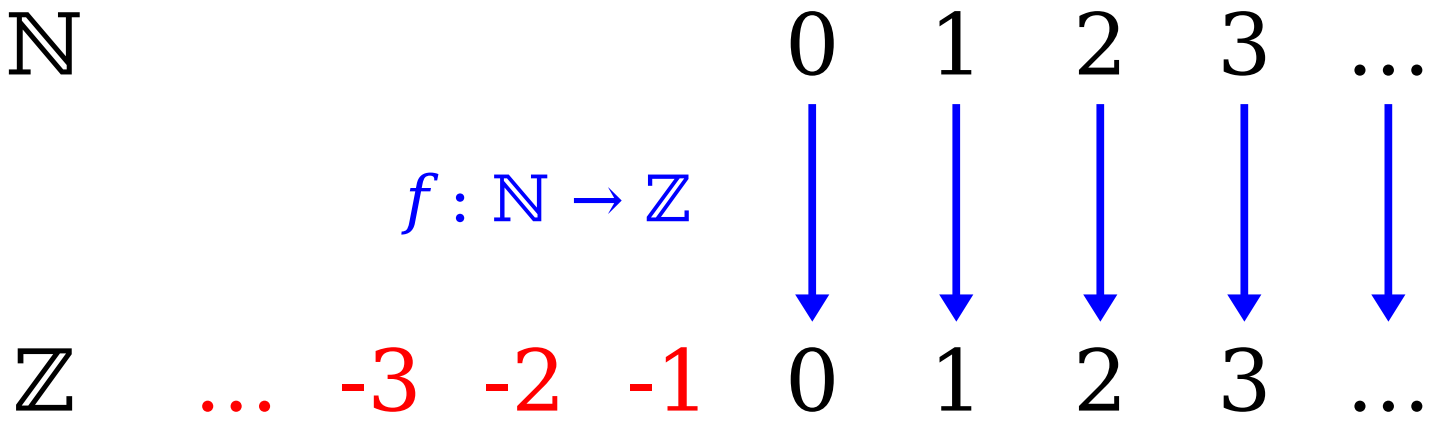
$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

That means that this function isn't a bijection, but it is injective.



$|S| \leq |T|$ if the elements of S and T can be paired up in a way that uses all the elements of S, but (not necessarily) all the elements of T.

So it seems like our idea of "a pairing of elements that uses everything from the first set, but not necessarily the second" can be expressed as "an injective function from the first set to the second!"



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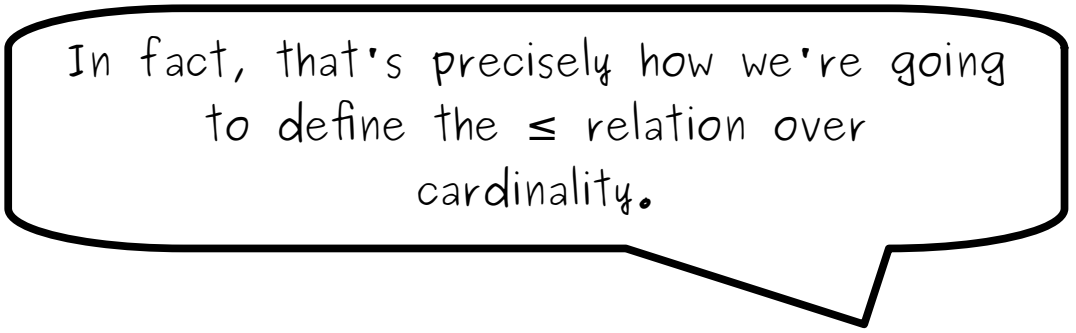
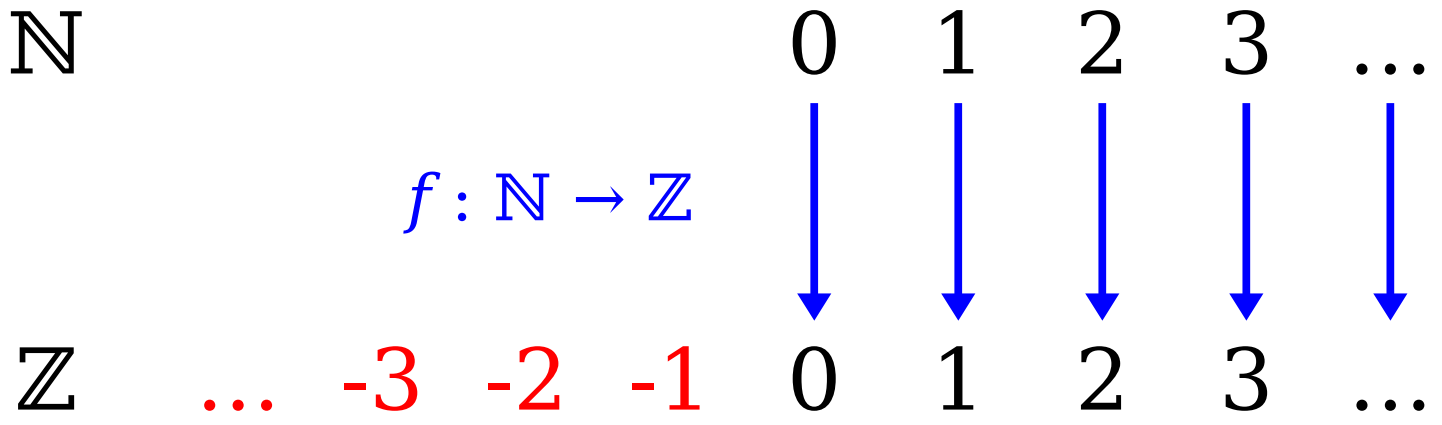
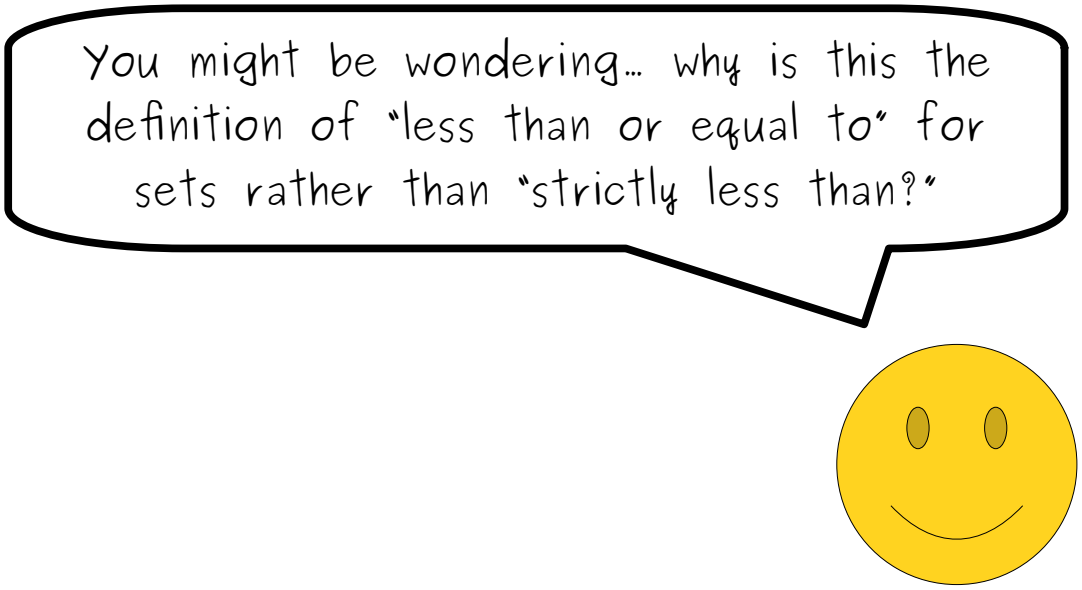




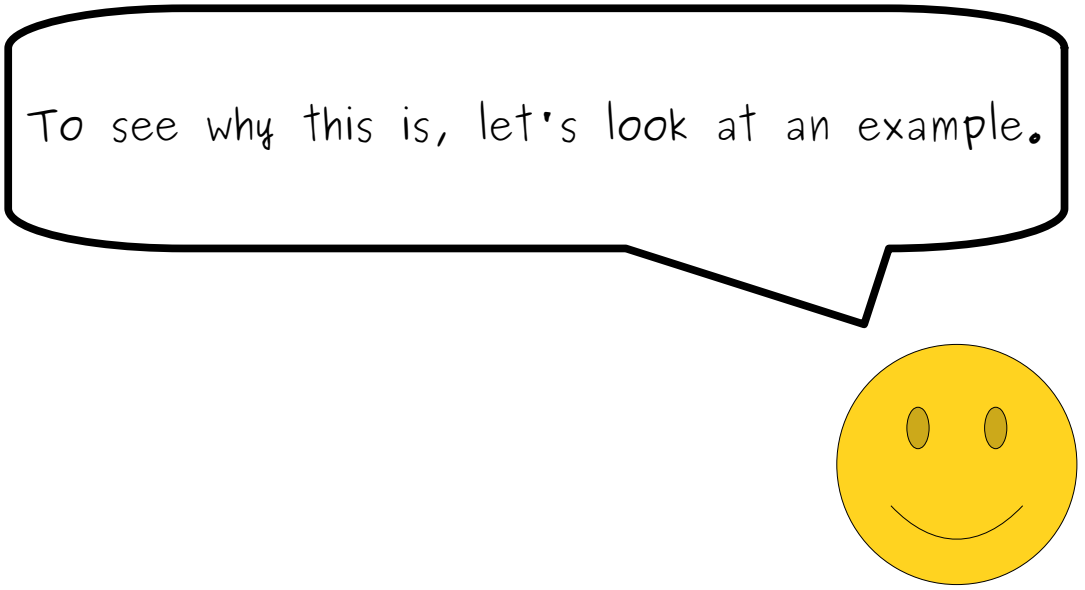
$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

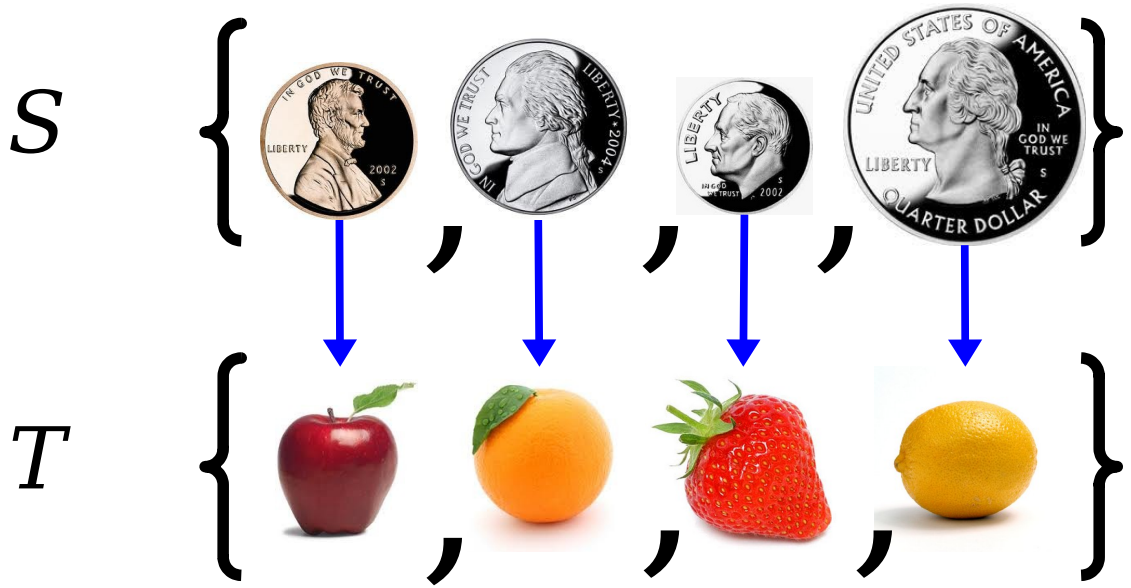
Here's the official definition of what it means for one set to be at least as large as another.

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$



$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

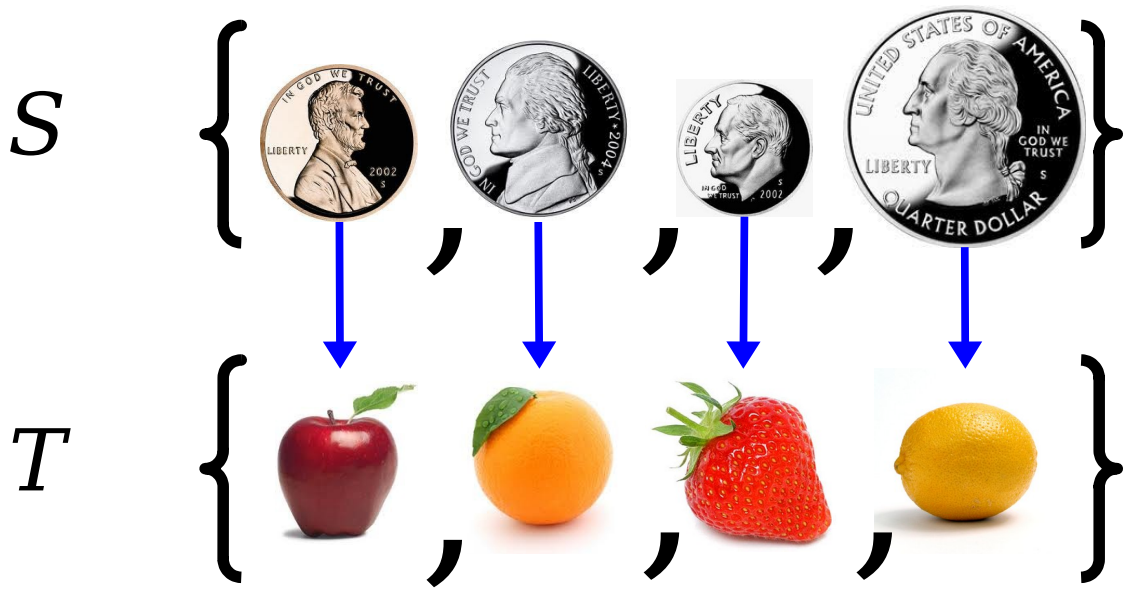




$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

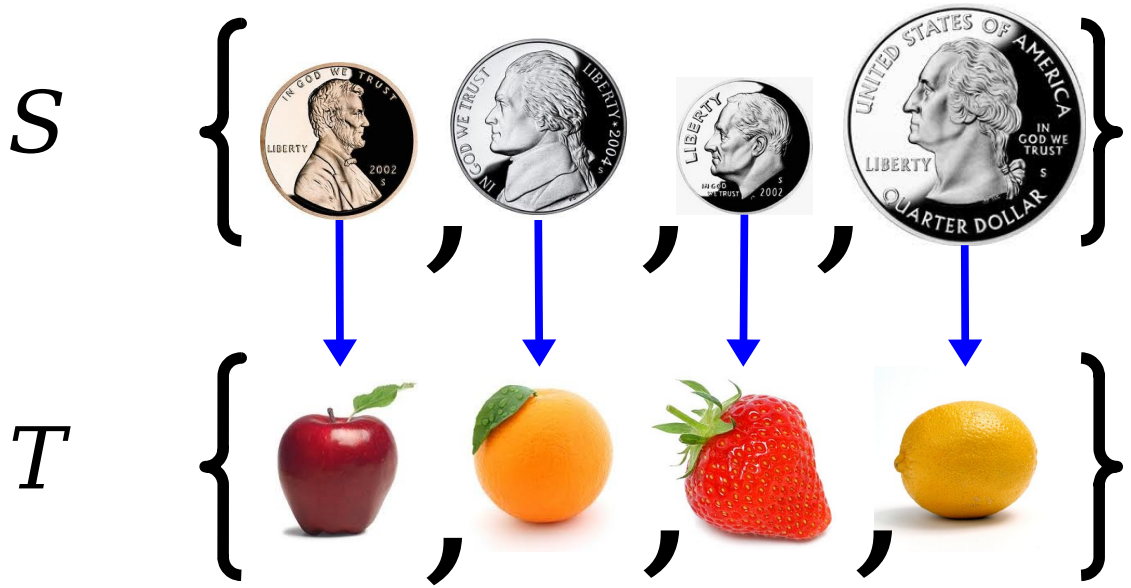
Here's two sets we've seen from before, along with a bijection between them.



$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

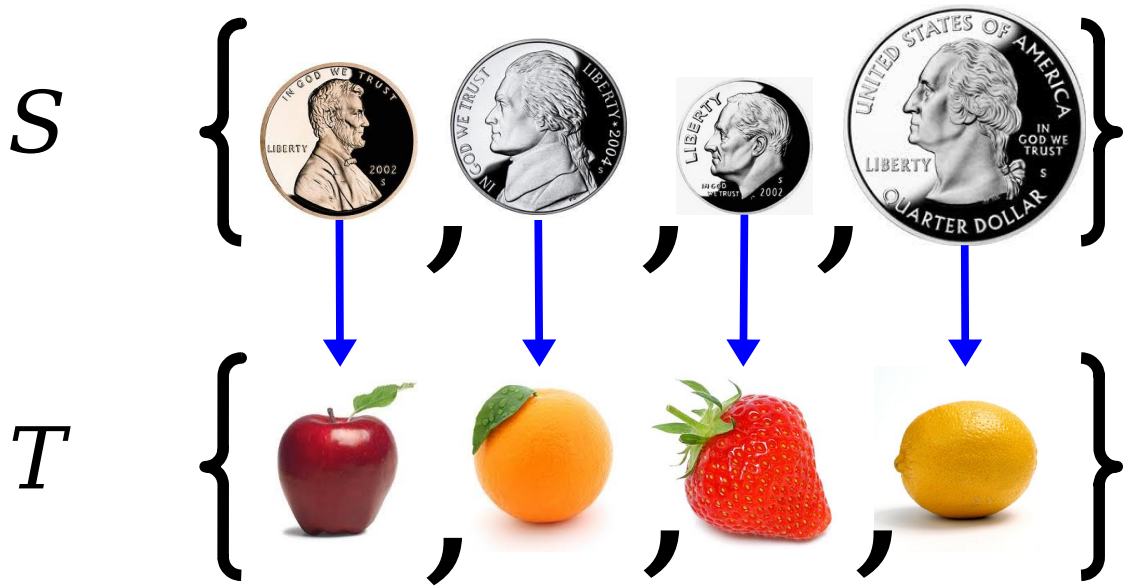
Because f is a bijection from S to T, we know that $|S| = |T|$.



$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

At the same time, this function f is also an injection (remember, all bijections are also injective!)



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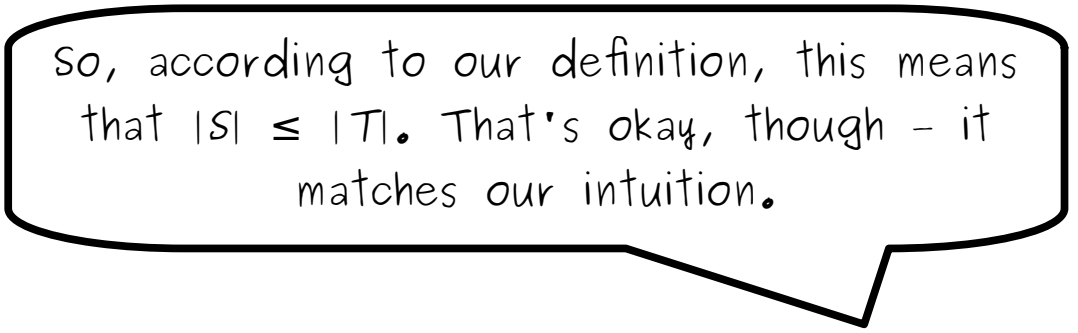
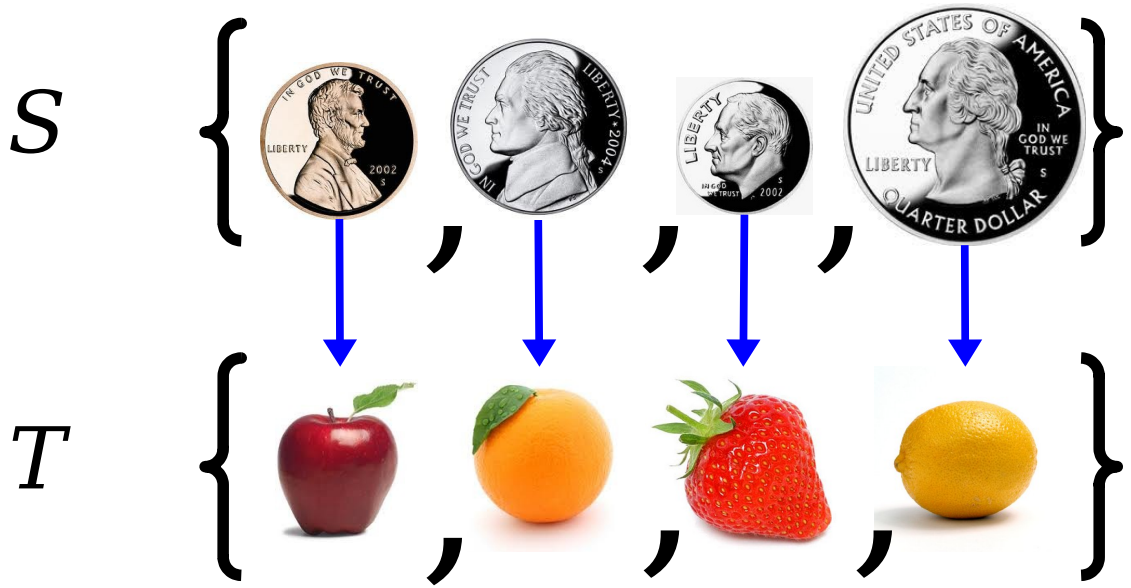


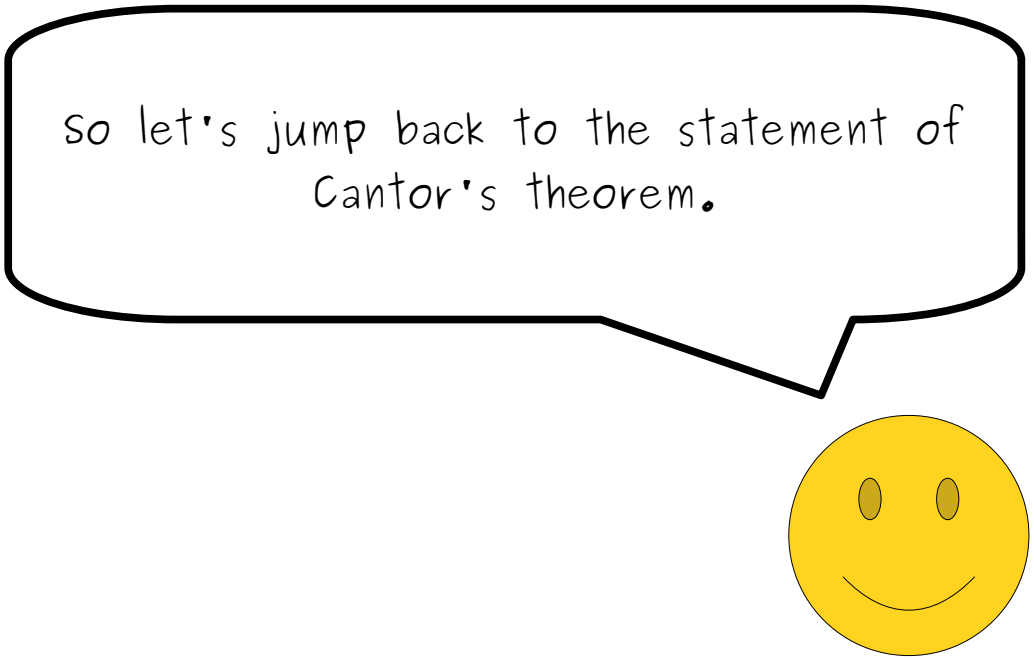


$f : S \to T$

$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

That's why we use this as a definition of $\leq$ over cardinalities – it's to be consistent with our definition of $=$ over cardinalities.

If S is a set, then $|S| < |\wp(S)|$



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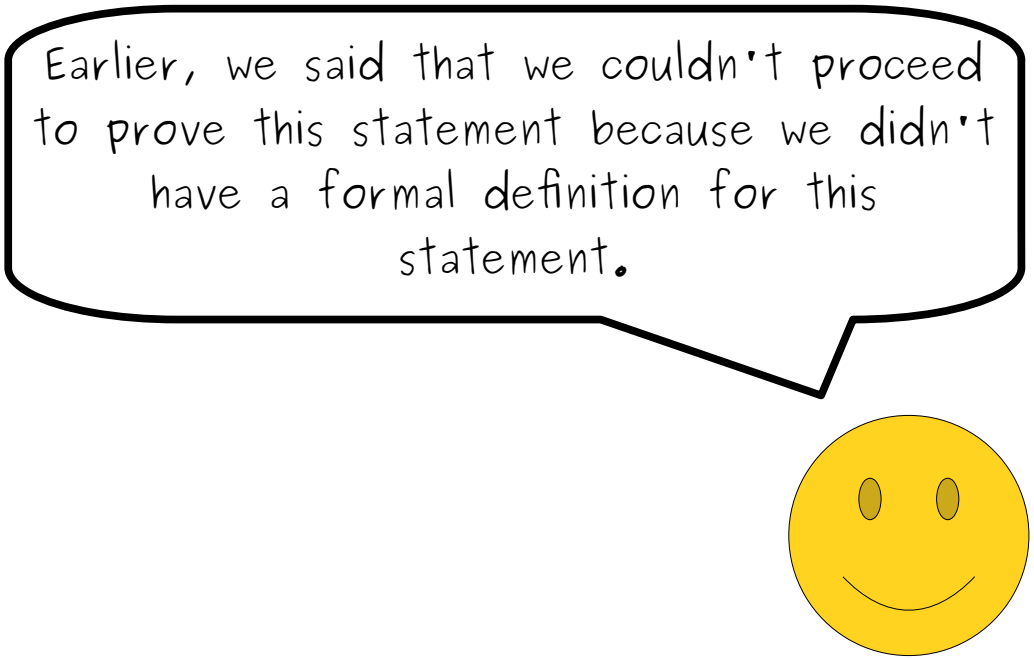

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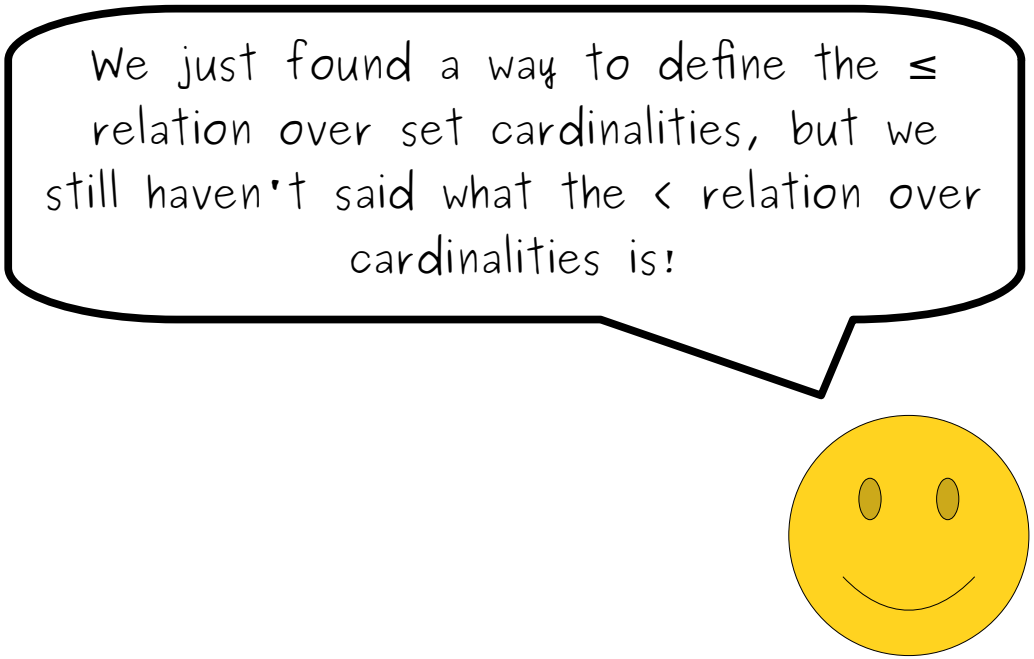

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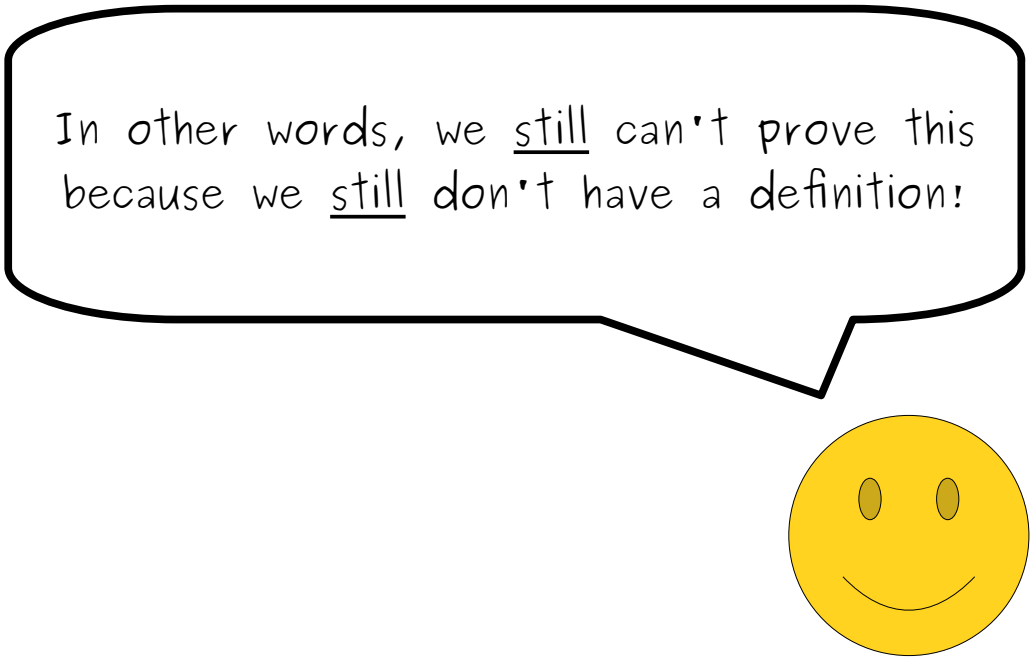

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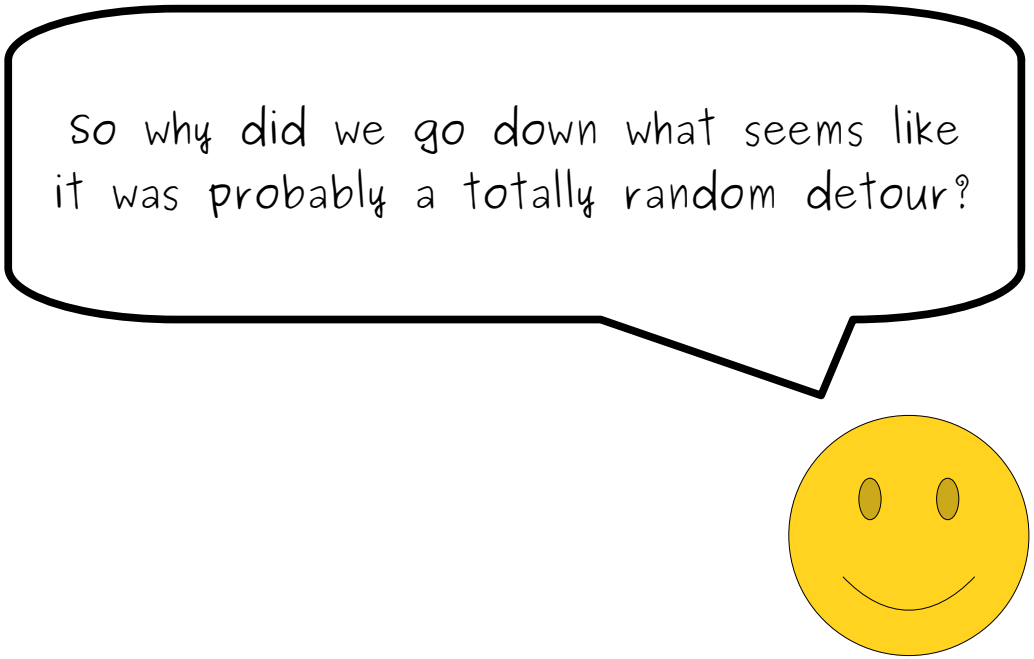

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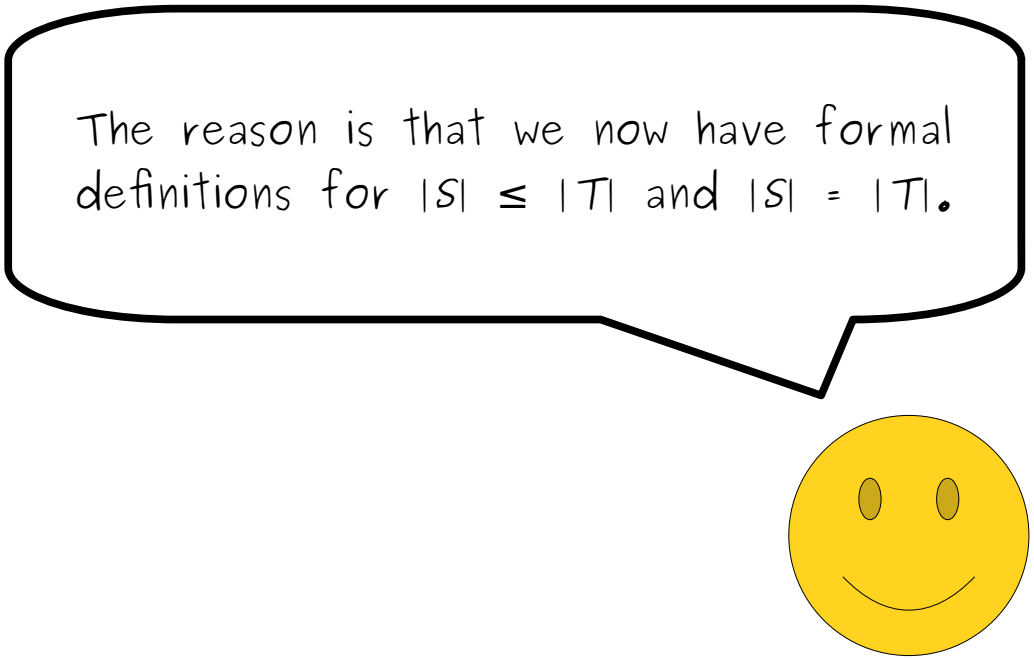

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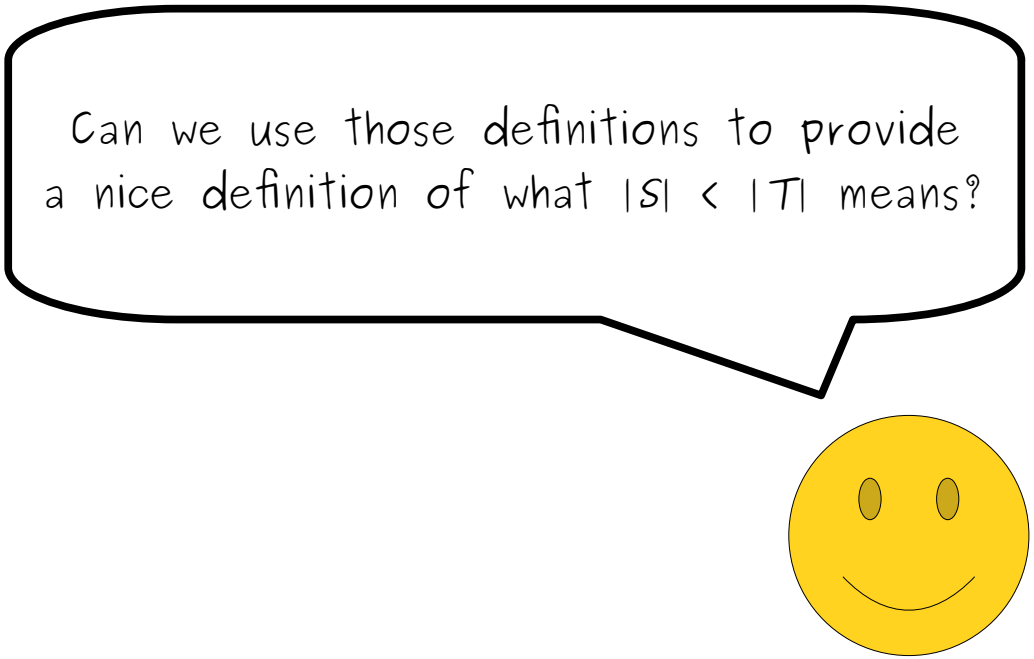
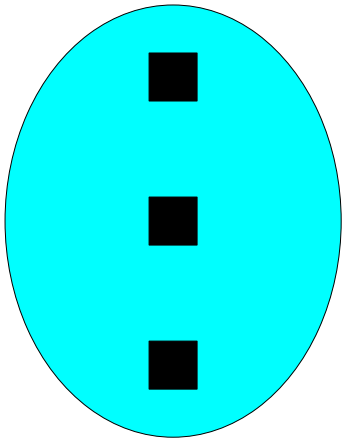
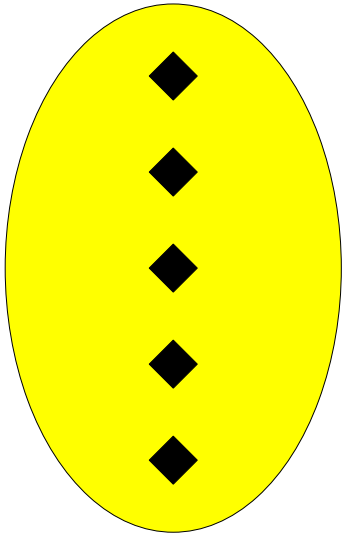
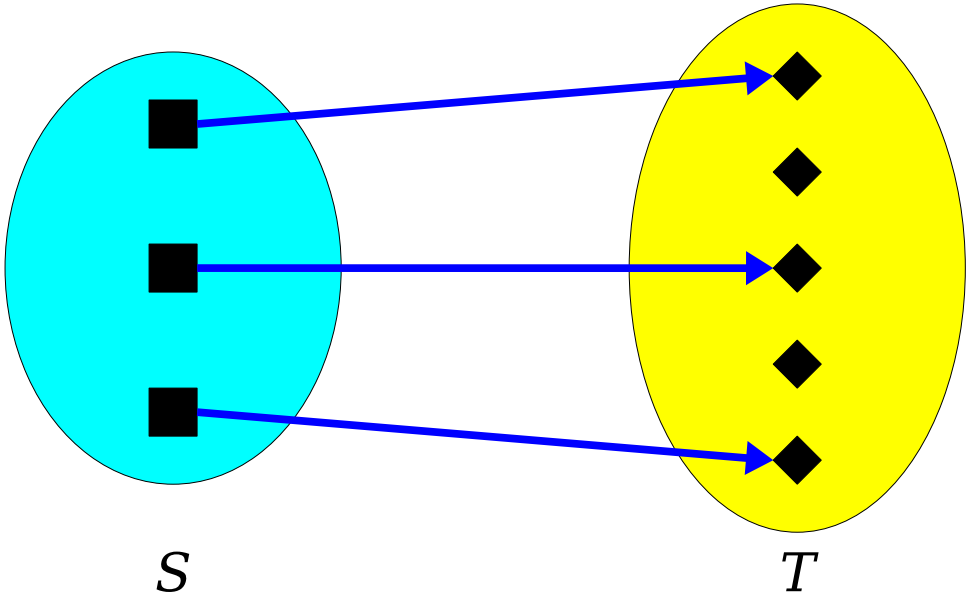




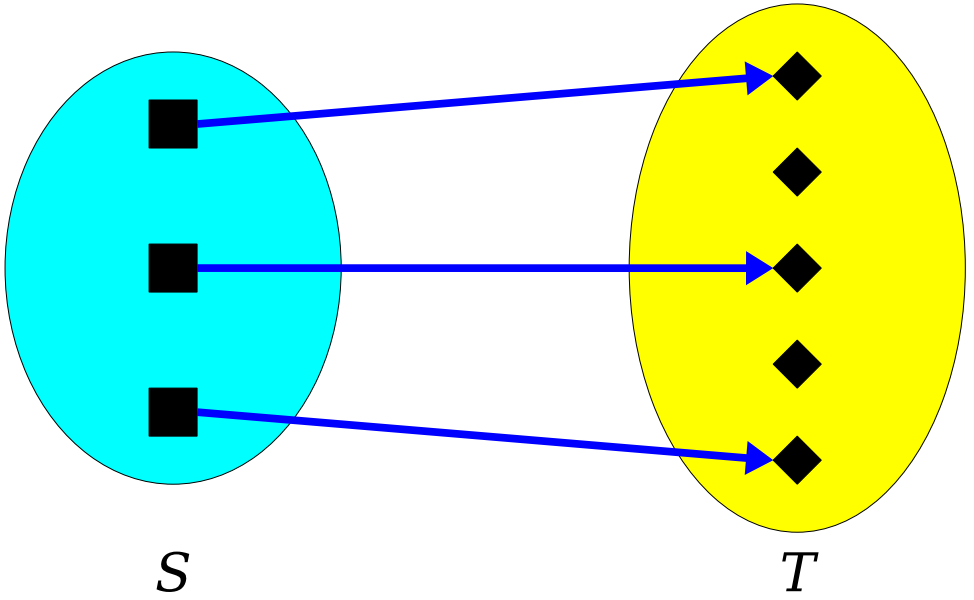
S



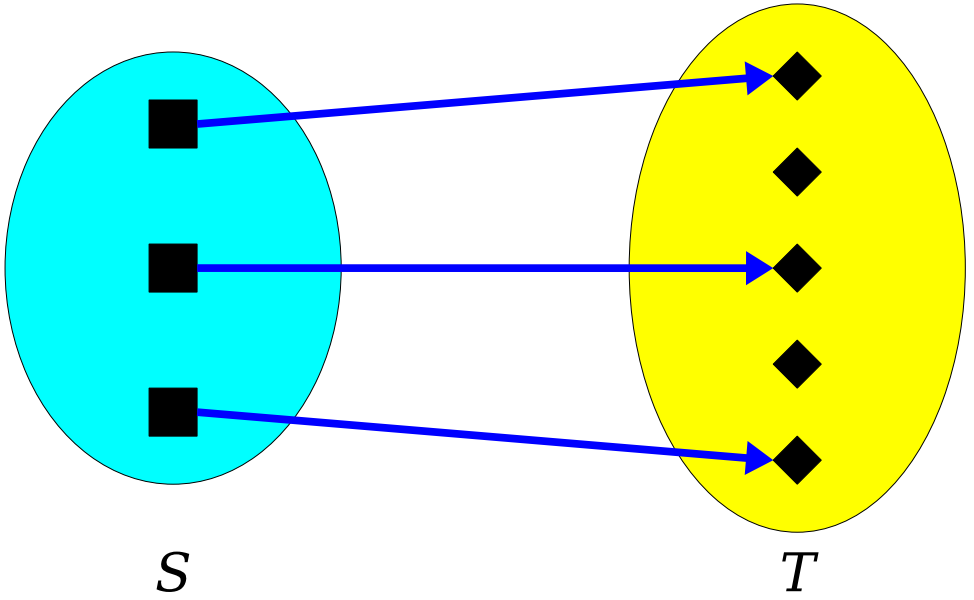
T



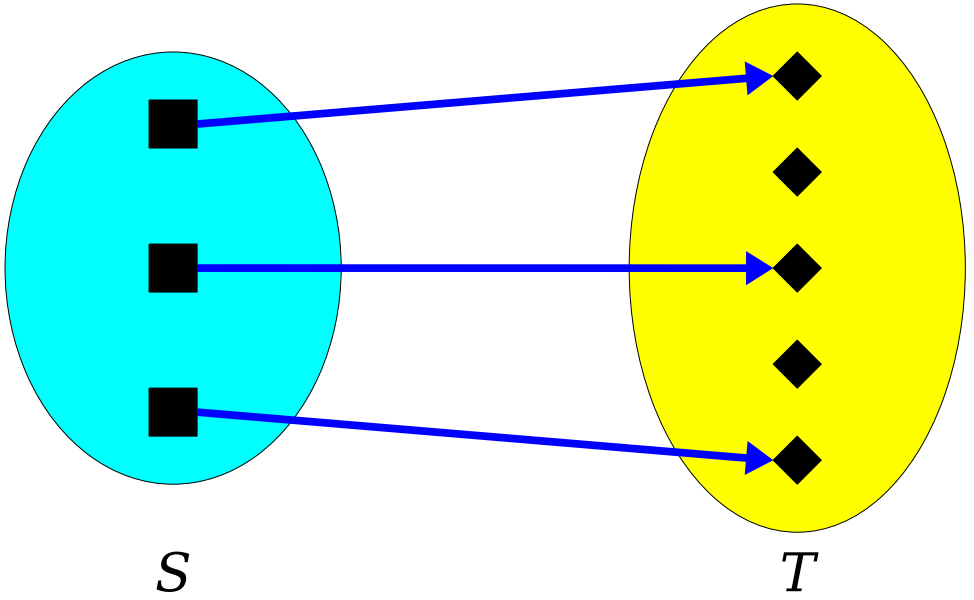
First, notice that $|S| \leq |T|$, since we can find an injection from S to T.



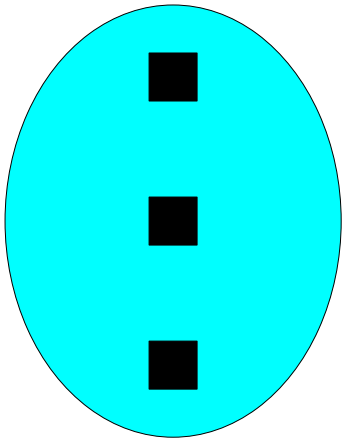
We can also see that $|S| \neq |T|$, since no matter how hard we try, we'll never find a bijection between the two sets.



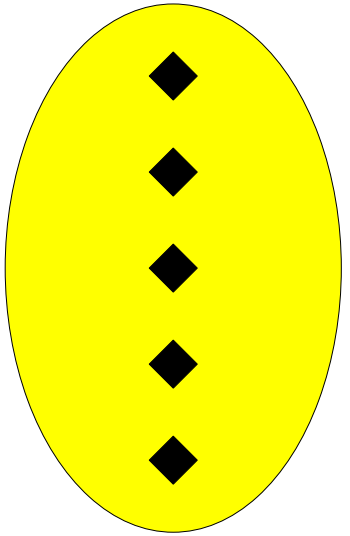
So we've seen that $|S| \leq |T|$ and that $|S| \neq |T|$...



…and intuitively it seems like the set S is strictly smaller than the set T.



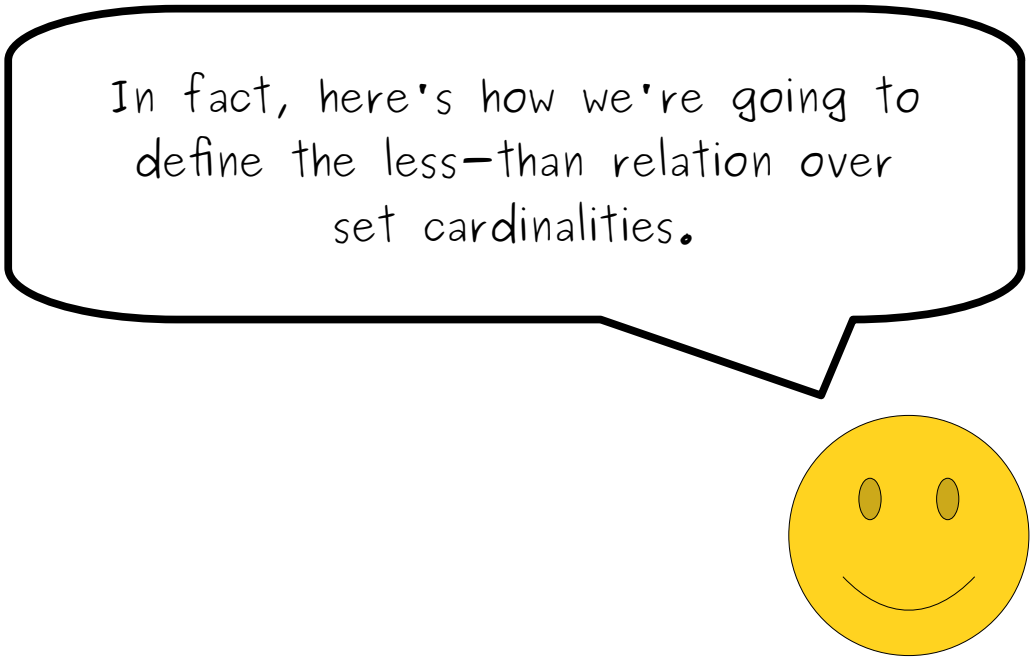
S



T

Perhaps that observation would make for a good definition of less-than over cardinalities!

$$|S| < |T| \quad \text{if} \quad |S| \leq |T| \text{ and } |S| \neq |T|$$



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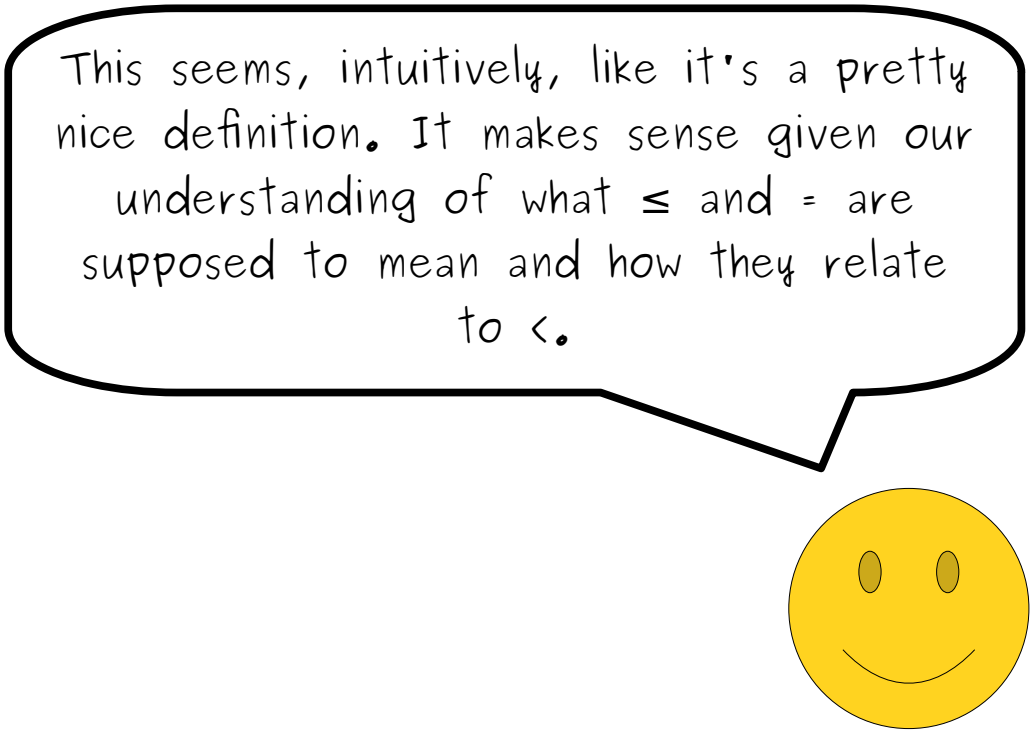


$|S| < |T|$ **if** $|S| \leq |T|$ **and** $|S| \neq |T|$

But remember that the ≤ and ≠ relations over cardinalities are defined in terms of functions between sets. Let's see what this definition says when we expand this out a bit.

$|S| < |T|$ **if** $|S| \leq |T|$ **and** $|S| \neq |T|$

There is an injective function $f : S \to T$

This part of the definition says that there should be an injective function from S to T...

$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

…and this part says that there aren't any bijective functions from S to T.

$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

So, if we want to prove that one set is strictly smaller than one another, we should proceed in two parts.

$|S| < |T|$ **if** $|S| \leq |T|$ **and** $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

First, we should find an injective function from the smaller set to the larger one.

$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

Second, we should prove that there are no bijections from the smaller set to the larger one.

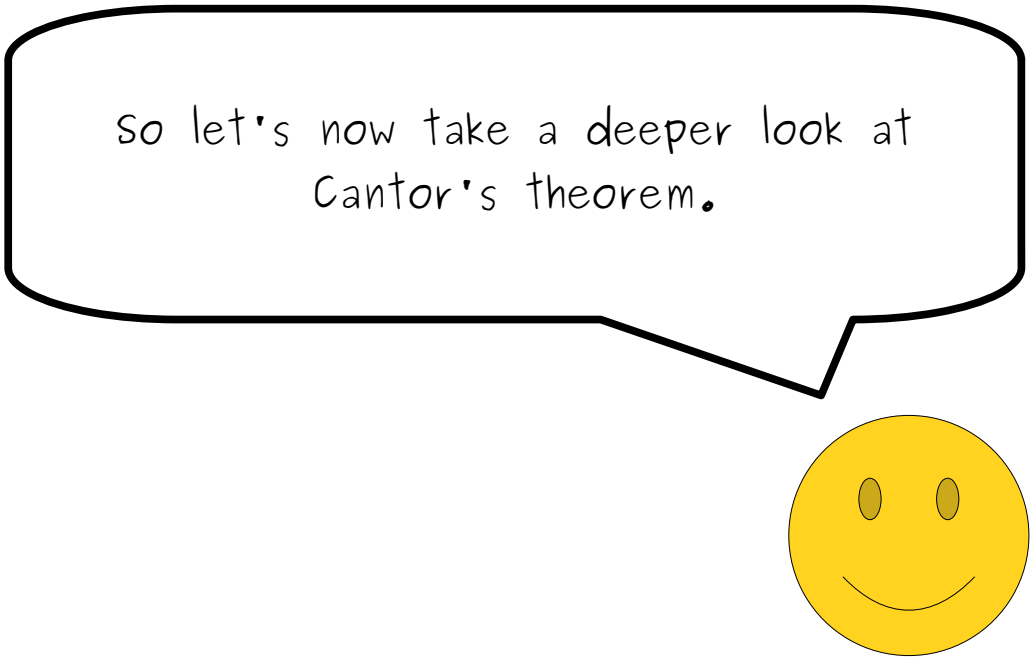
$|S| < |T|$ **if** $|S| \leq |T|$ **and** $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

If we can prove both of these statements, then we've shown everything that we've needed to show!

If S is a set, then $|S| < |\wp(S)|$



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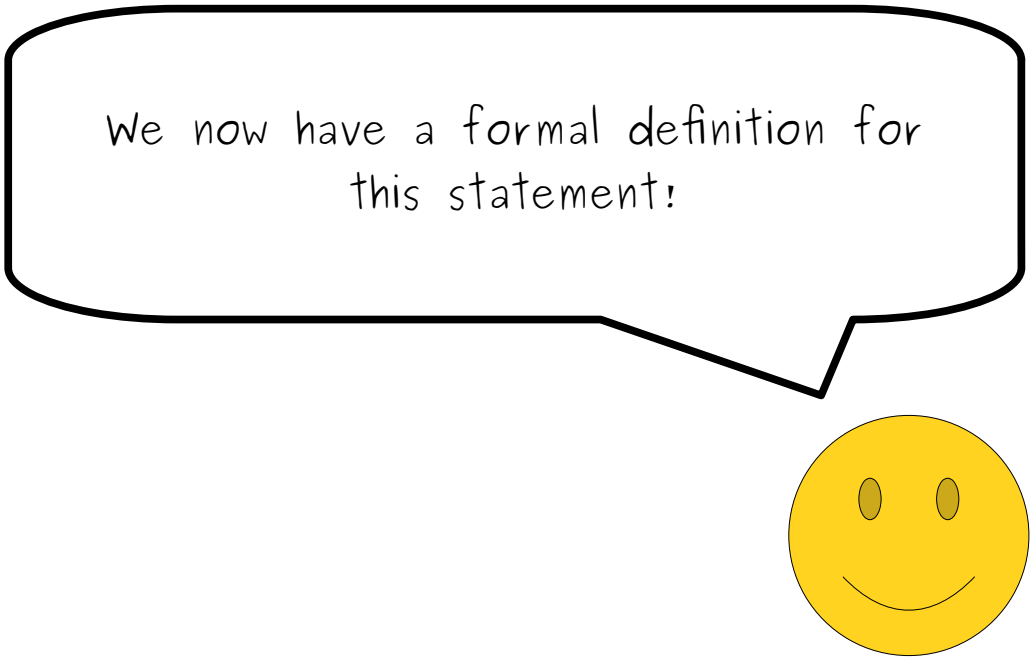


There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

We now know what we need to prove! We need to find an injection from S to $\wp(S)$ and to prove that there's no bijection from S to $\wp(S)$.

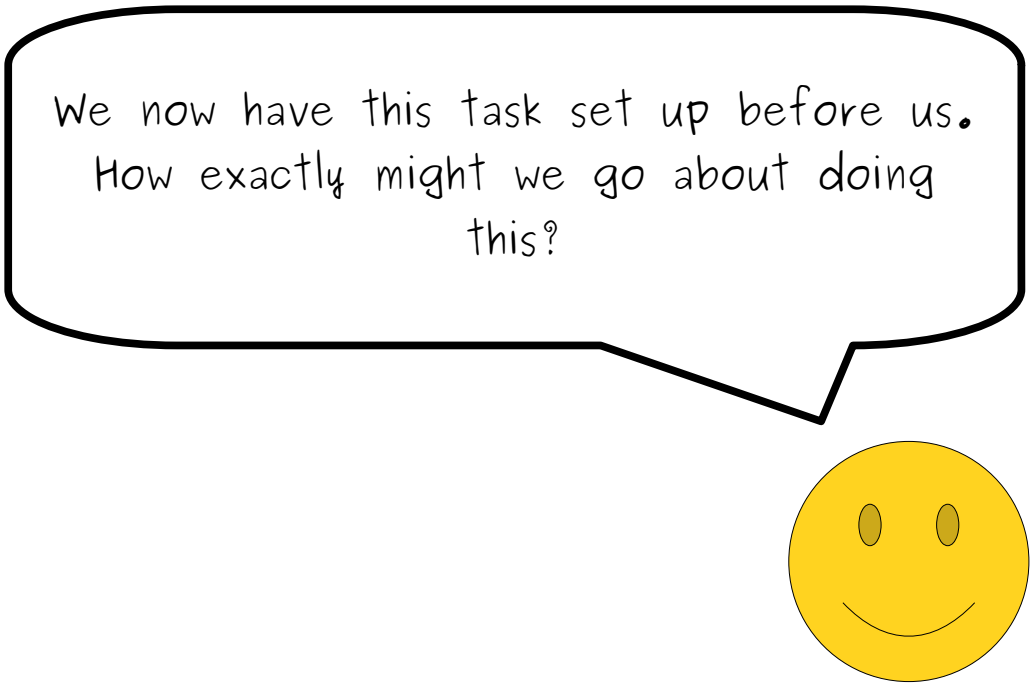
There *is* an injection $f : S \to \wp(S)$

There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

Let's start with this part. Turns out, it's not too bad!

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



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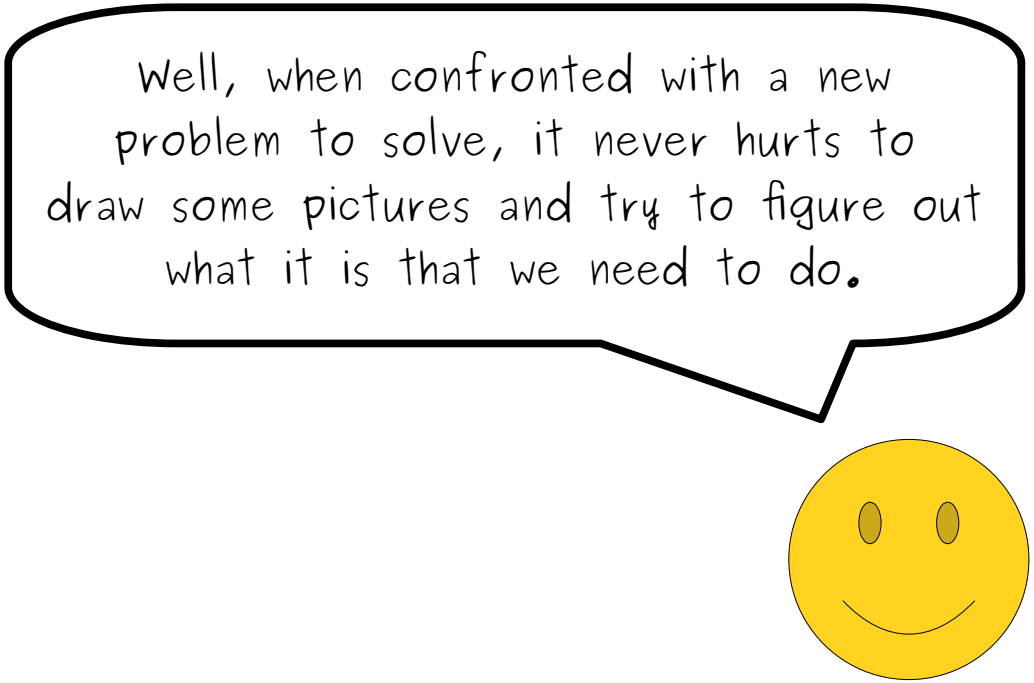

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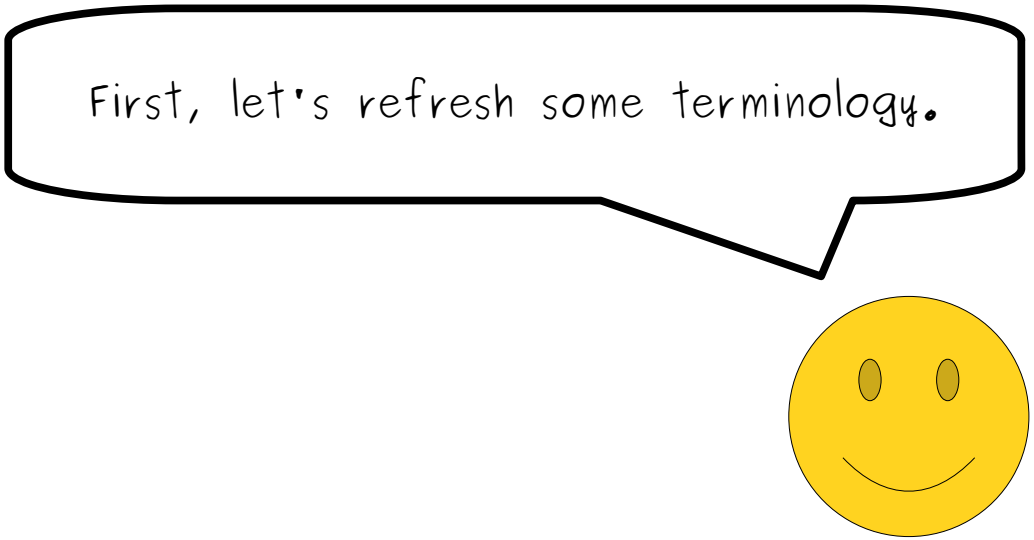

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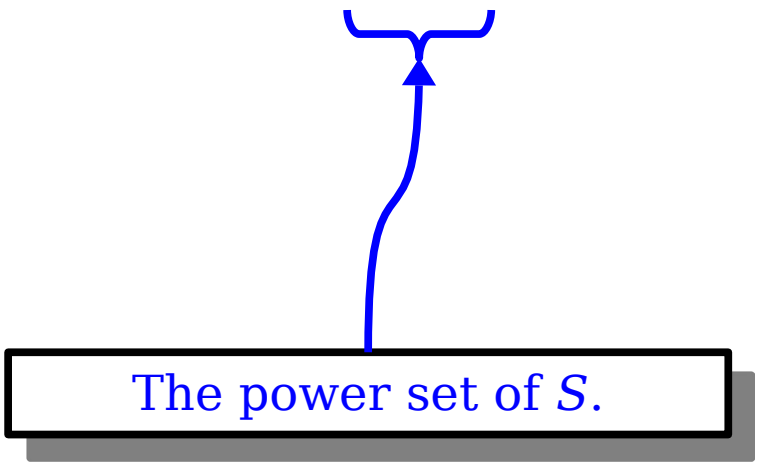
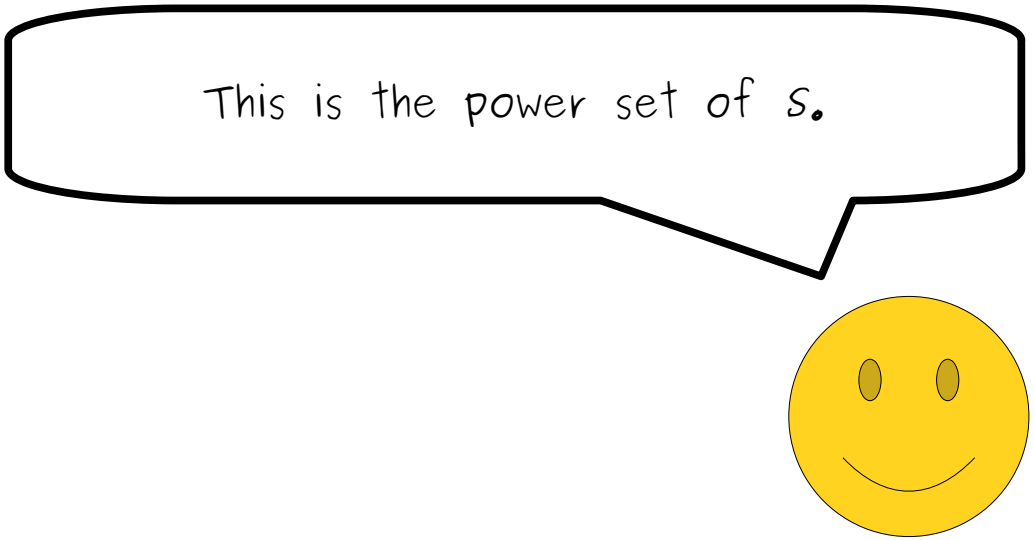
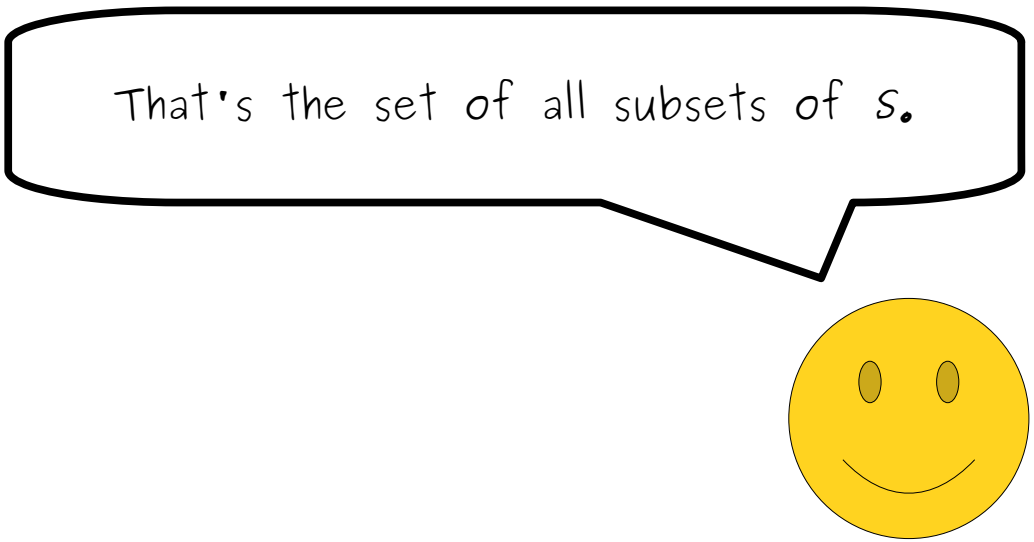




Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

The power set of S.
The set of all subsets of S.



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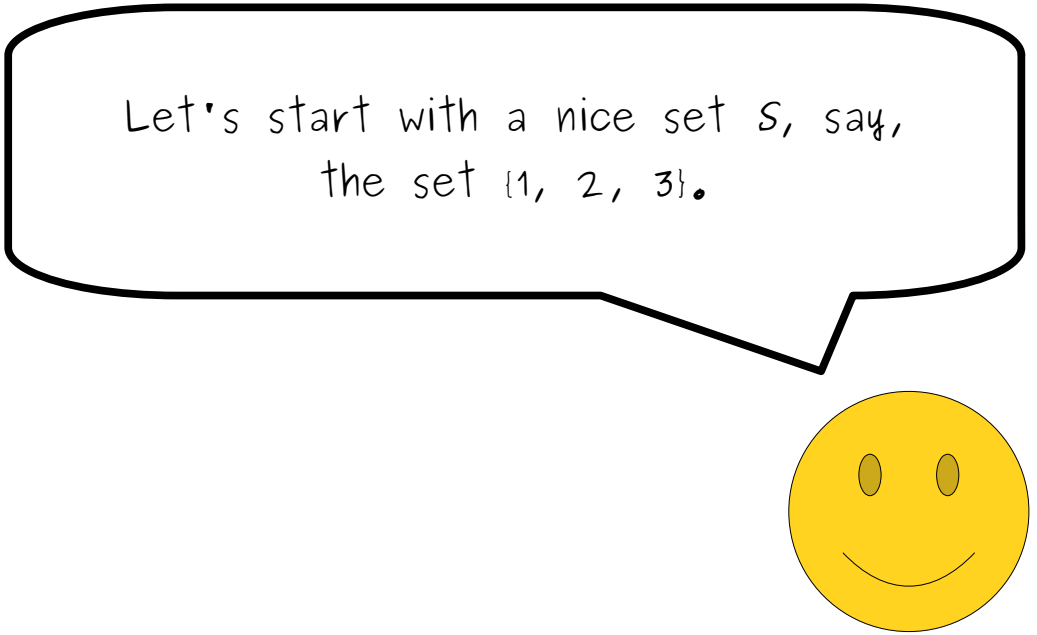
The power set of S.
The set of all subsets of S.

So if our goal is to find an injection from S to $\wp(S)$, we're looking for an injection from the set S to the set of all subsets of S.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



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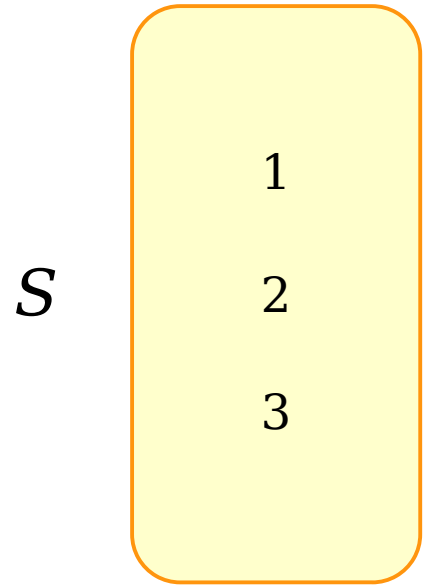

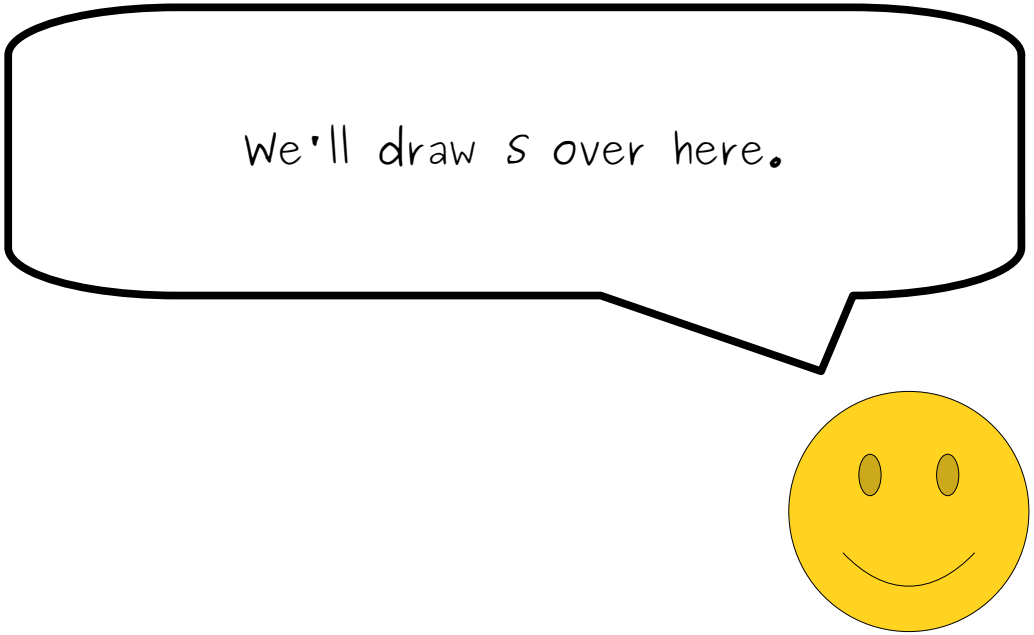

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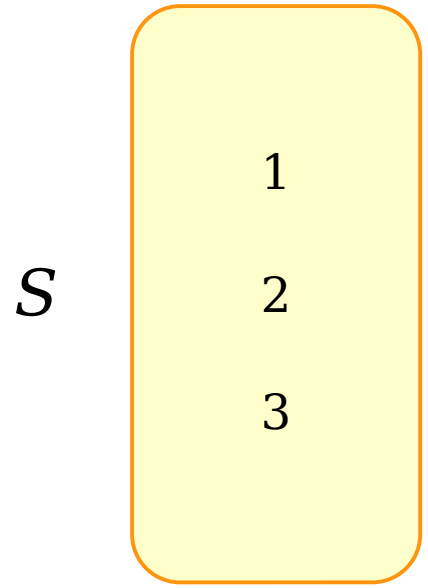

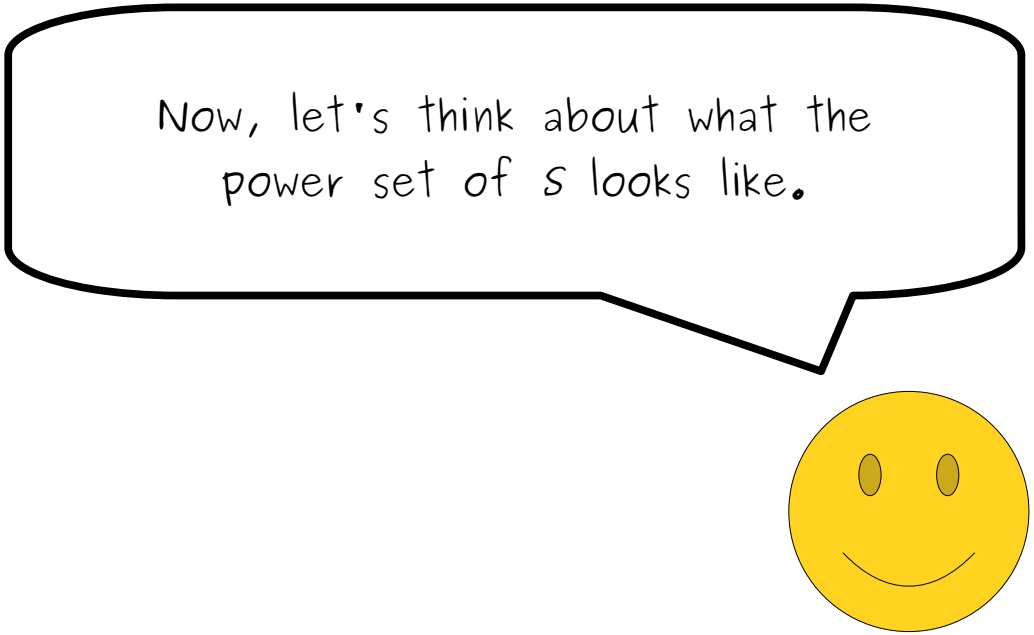

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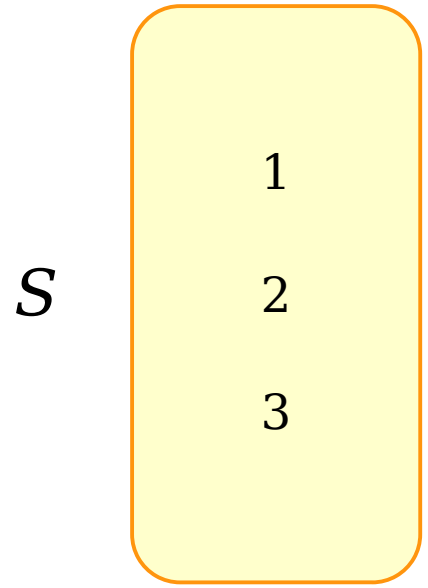

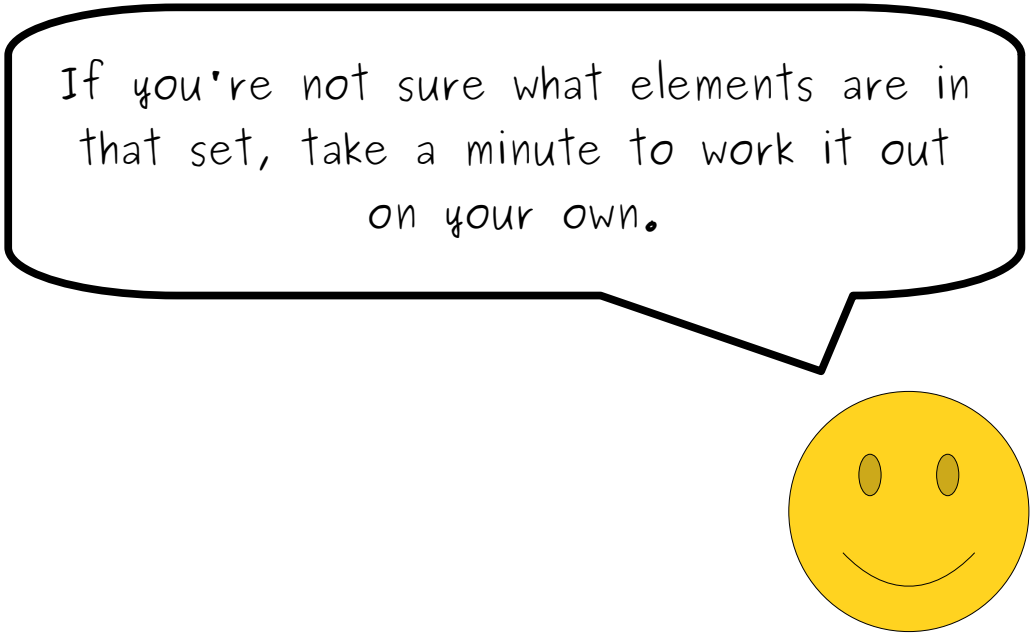

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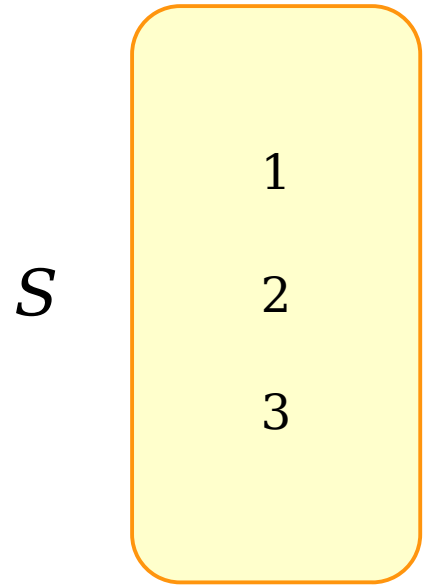

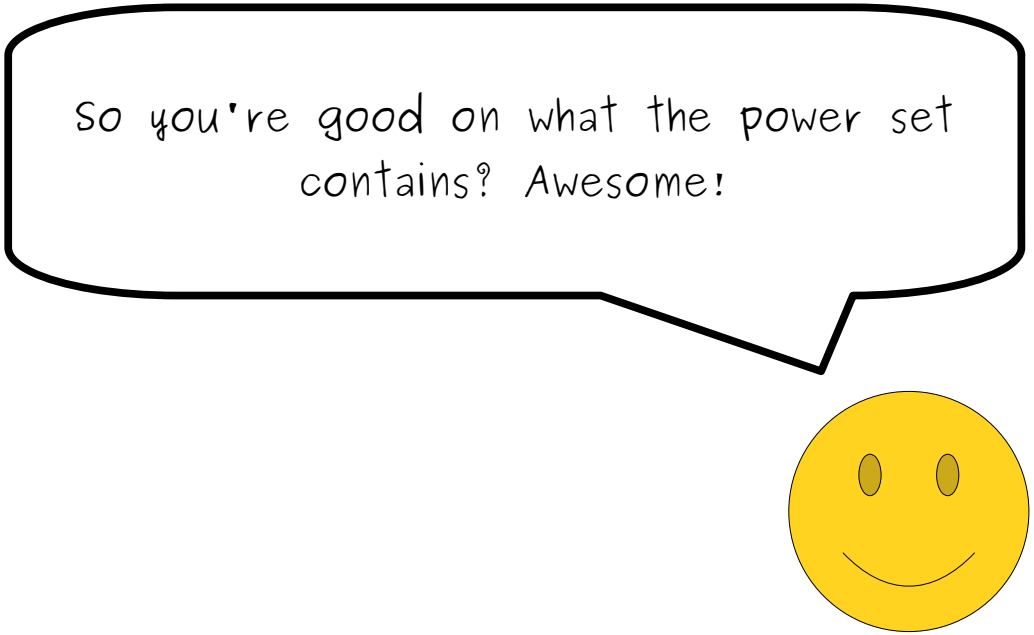

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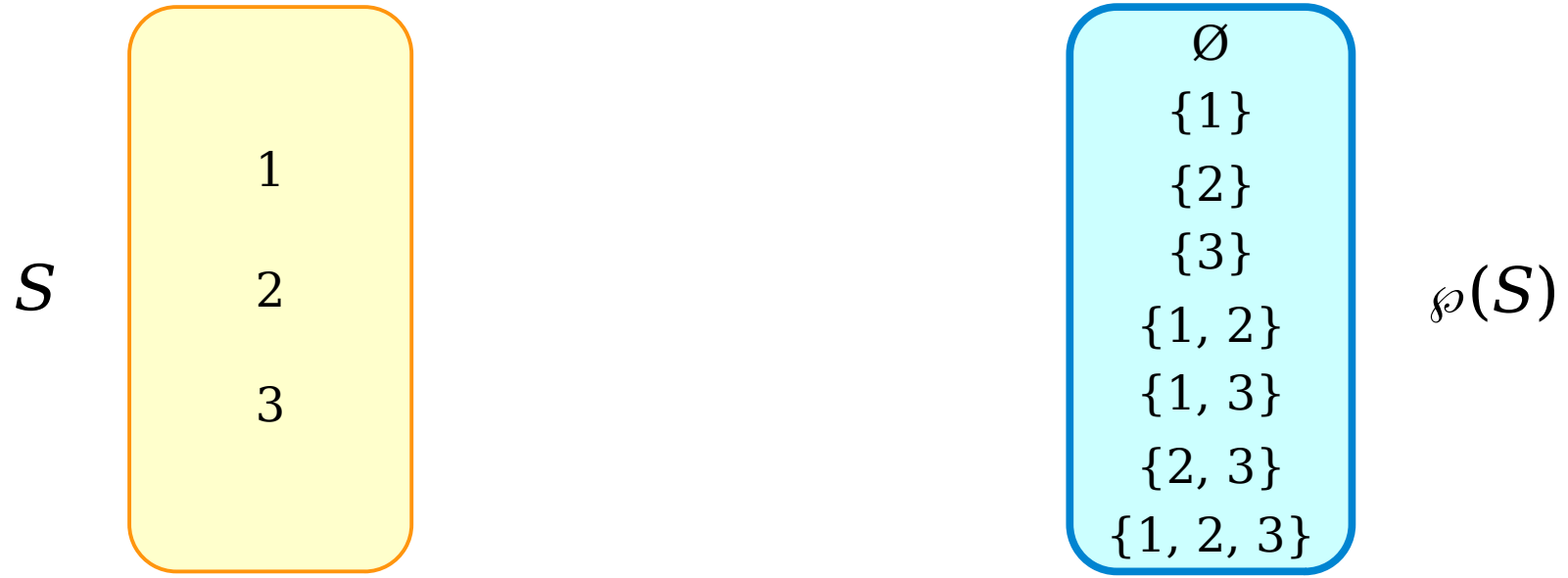
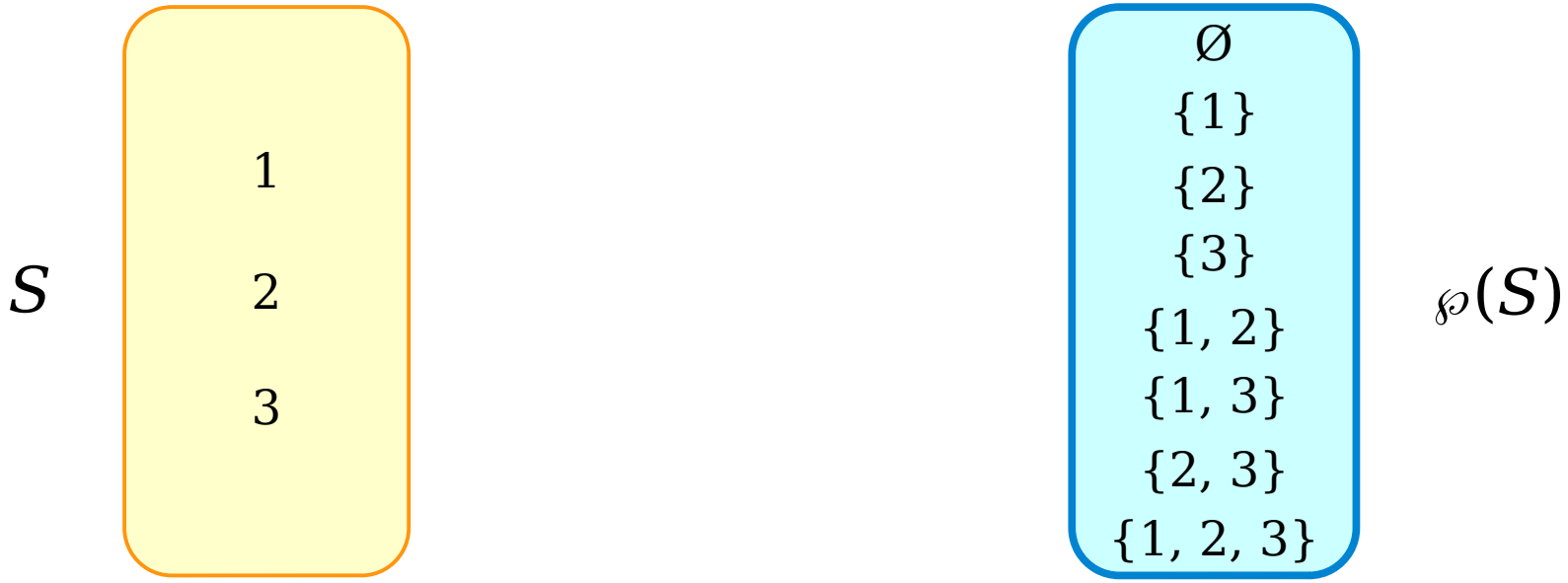


Here's what that looks like.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



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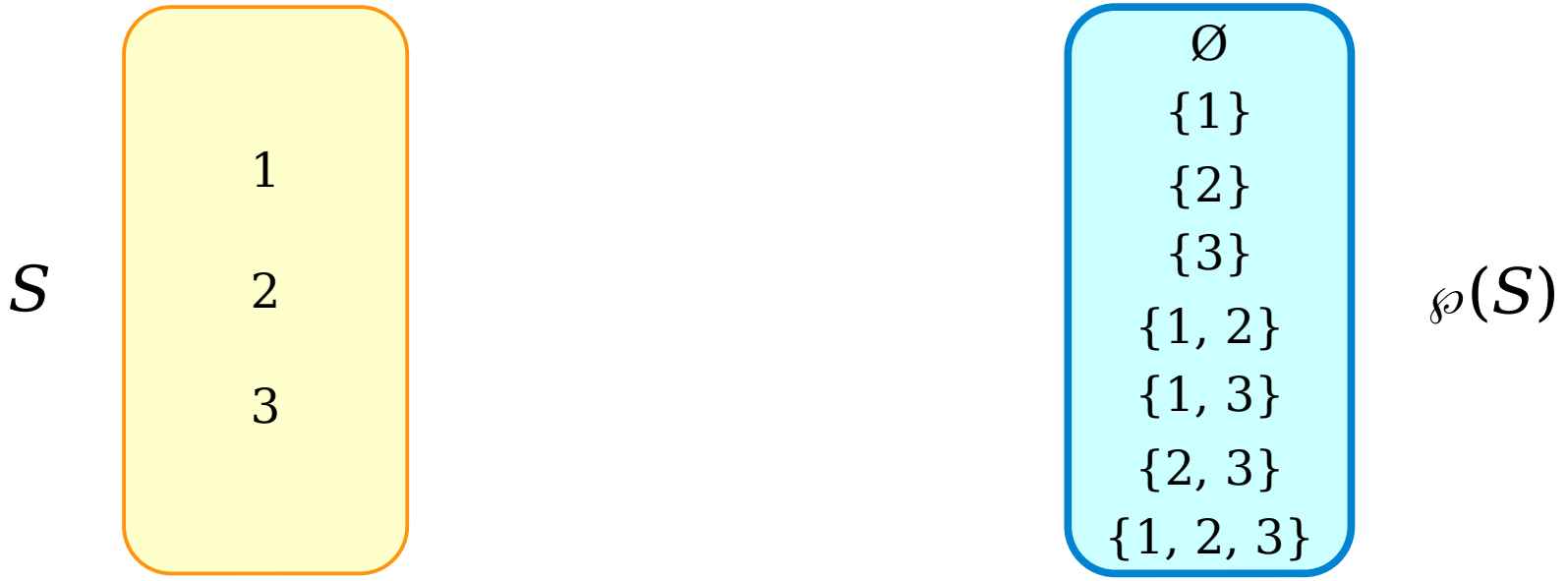
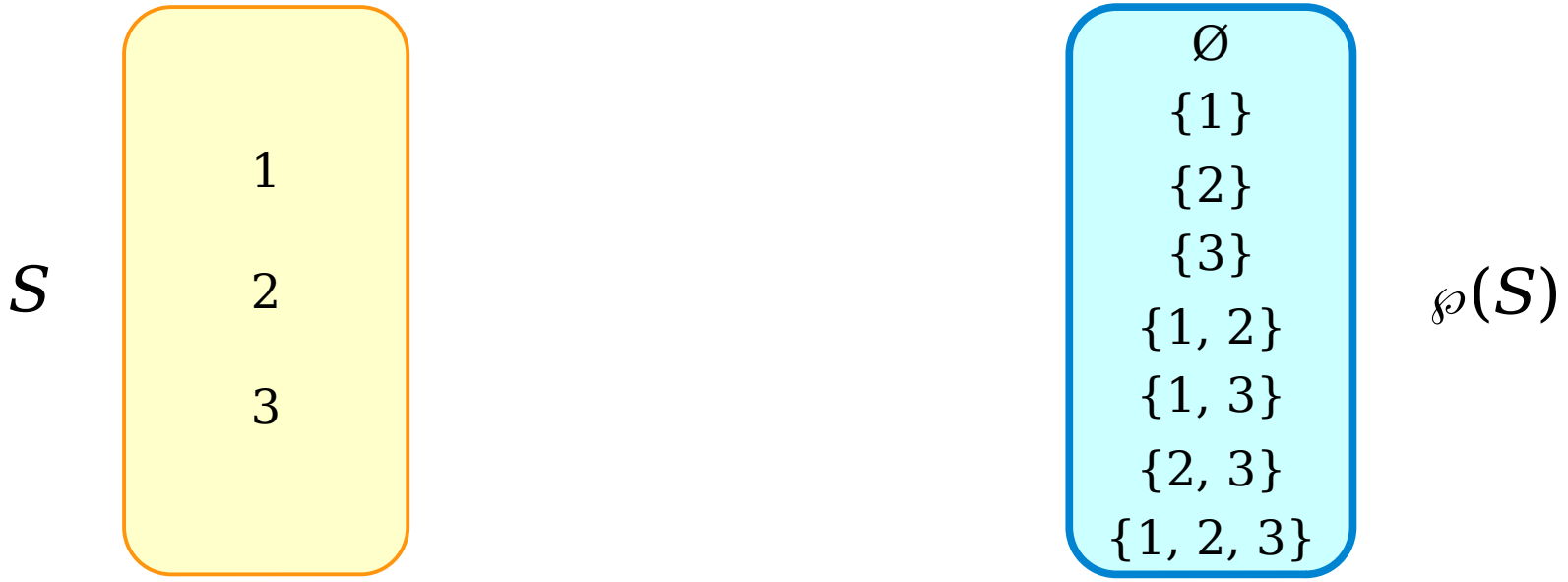


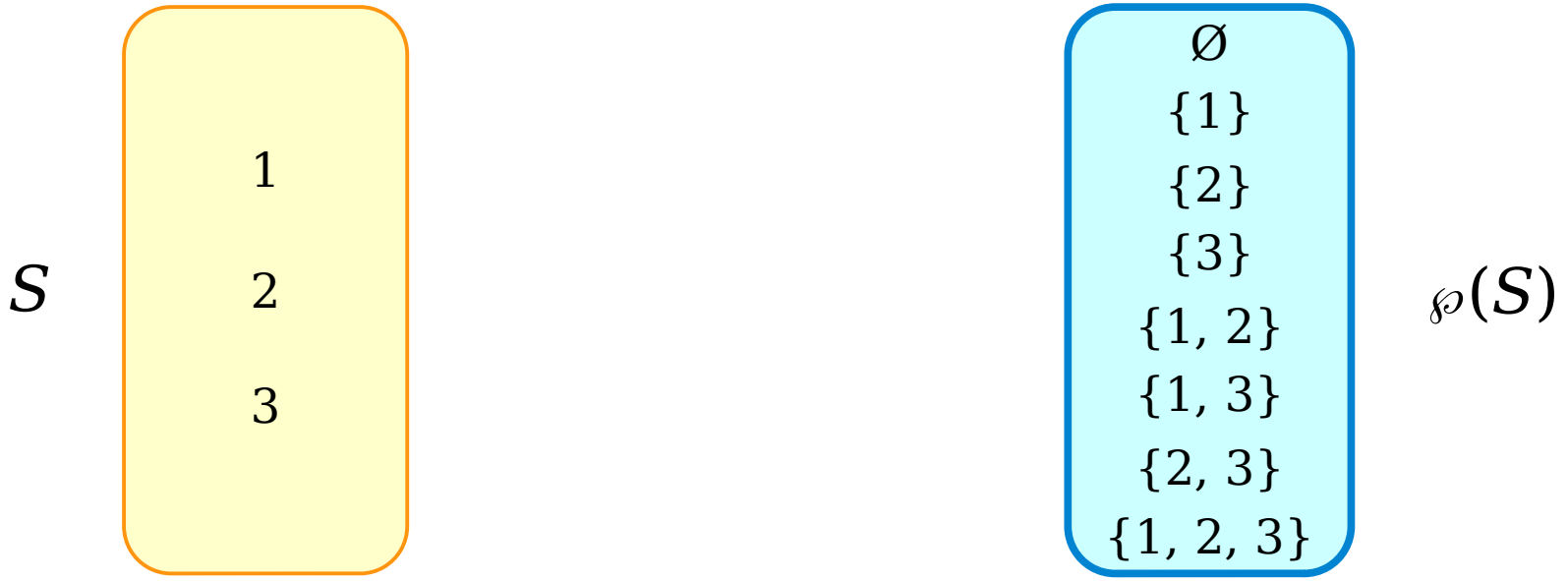
Remember our intuition from before – we want to pair off the elements of S with the elements of $\wp(S)$ so that we use up all the elements of S.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



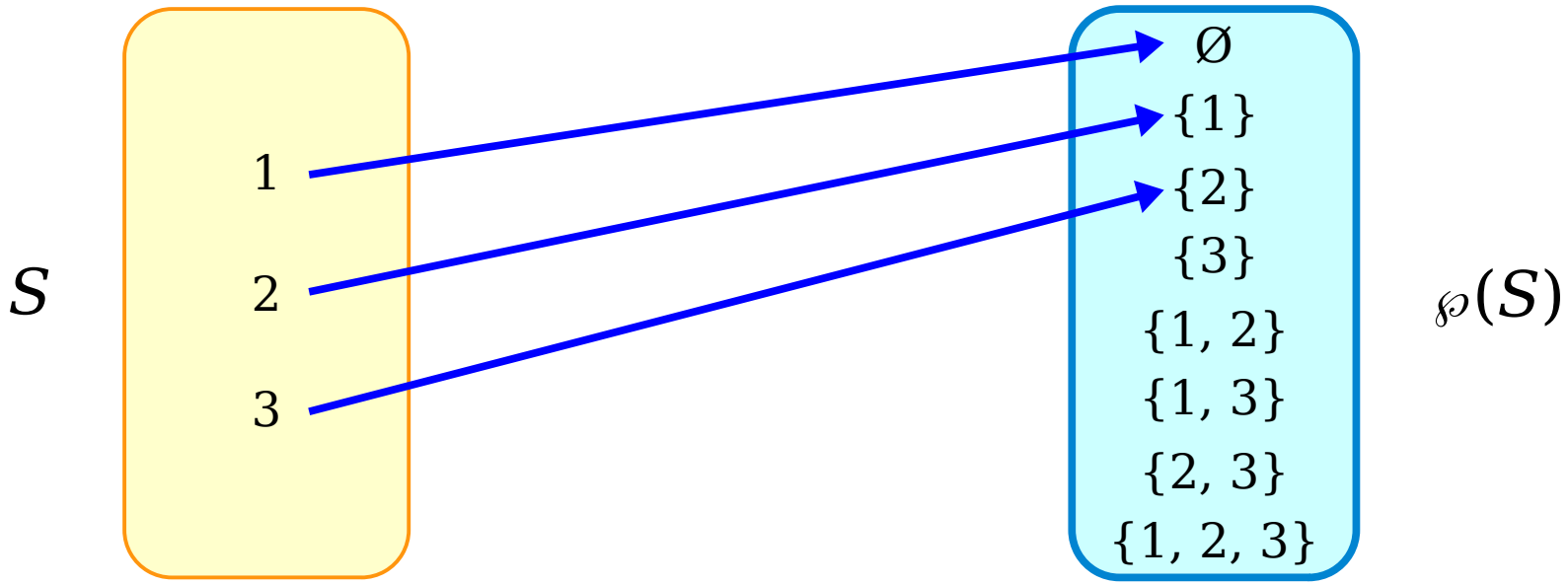
It's okay if some elements of $\wp(S)$ are left over – after all, we're ultimately trying to prove that $|S| < |\wp(S)|$!

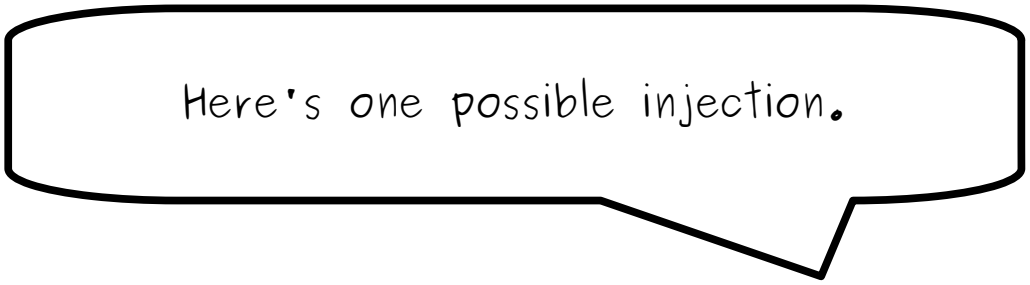
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



There are a lot of ways we could do this.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.





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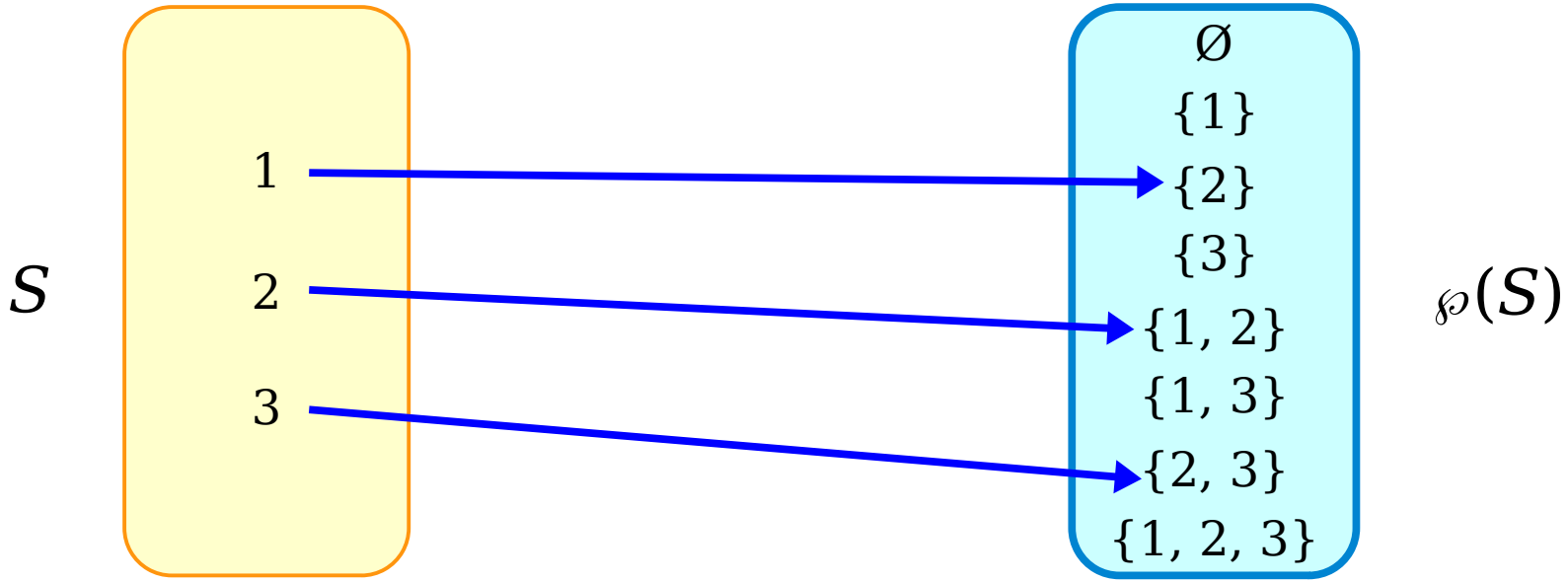

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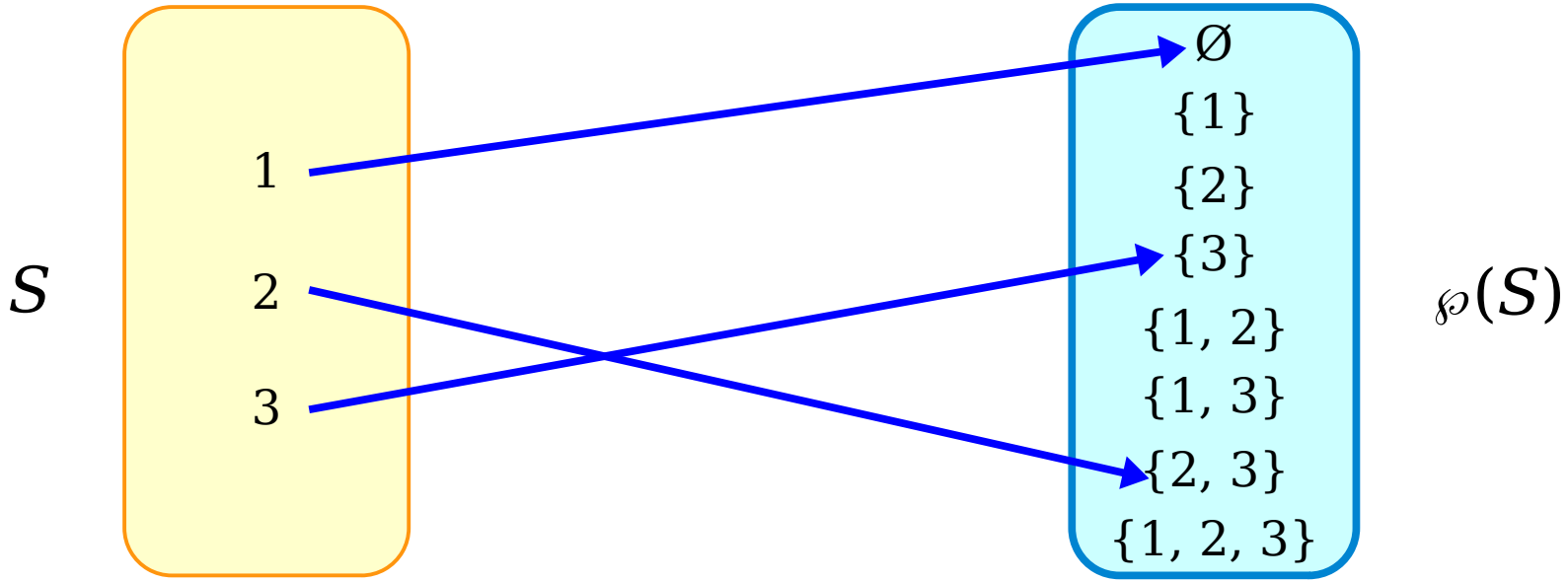

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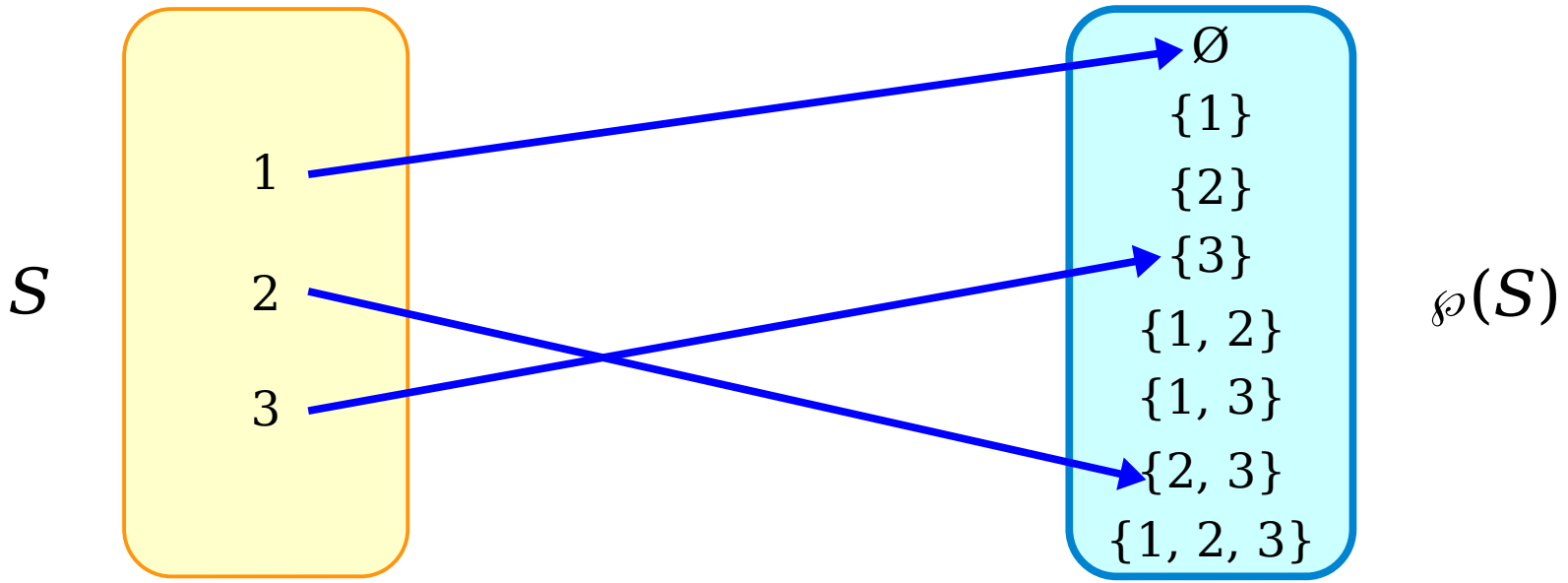
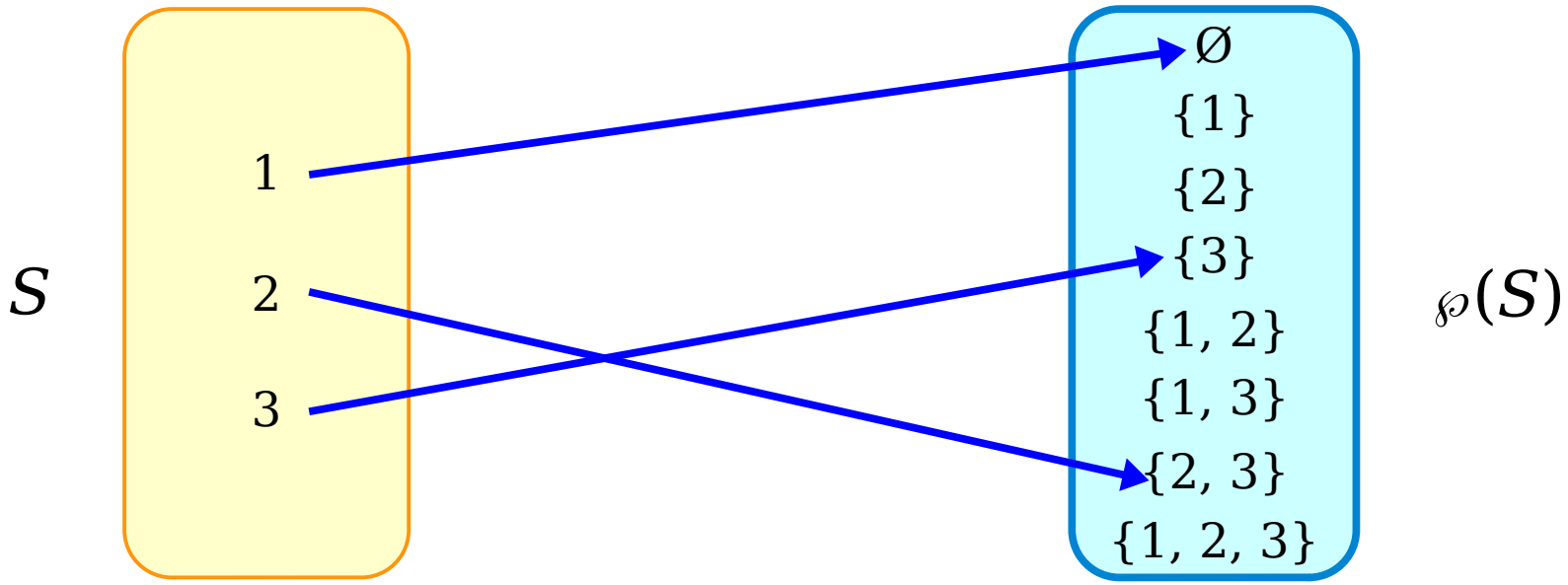


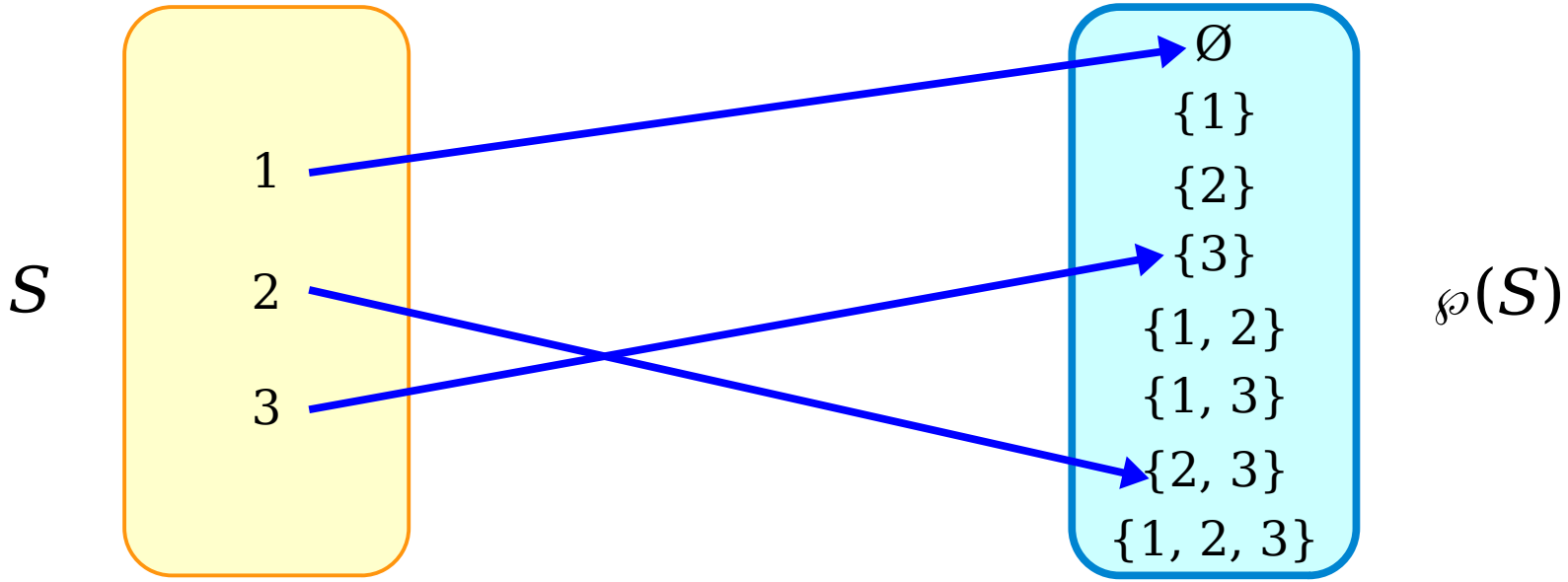
Now, ultimately, we're not trying to show that for this particular choice of S that an injection like this exists.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



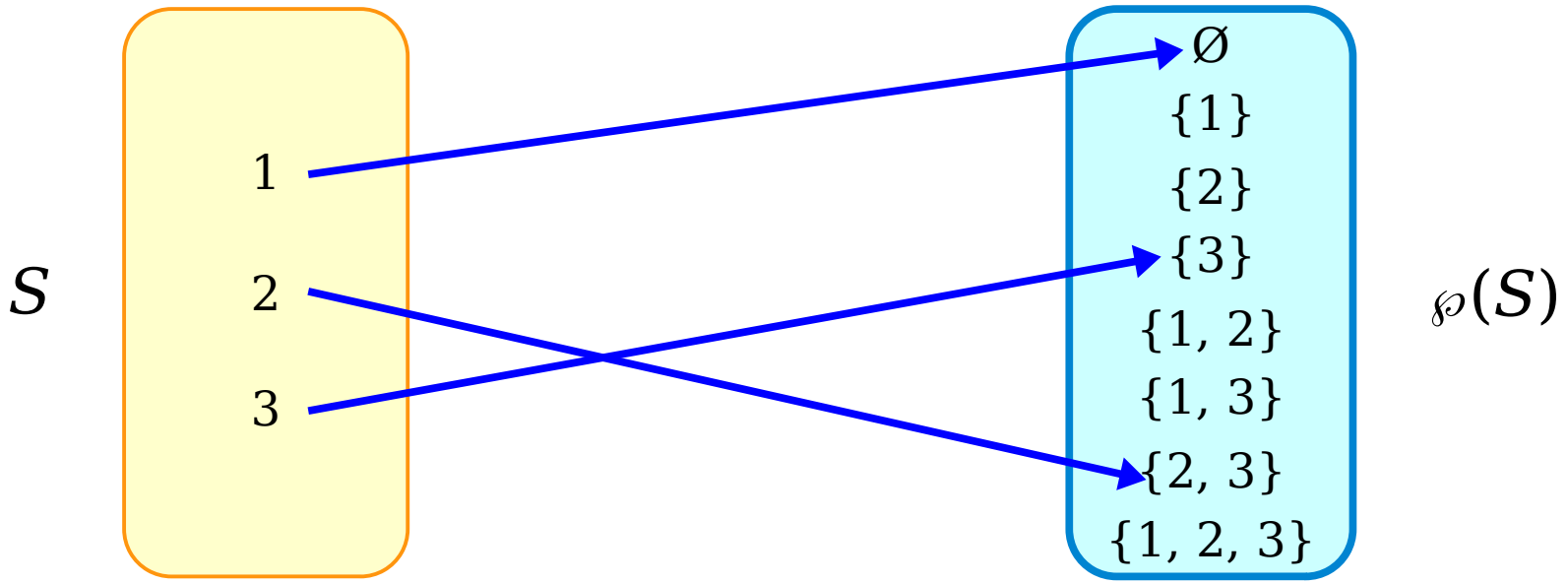
Rather, we're trying to show that no matter what set S we pick, we can always find an injection from S to $\wp(S)$.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



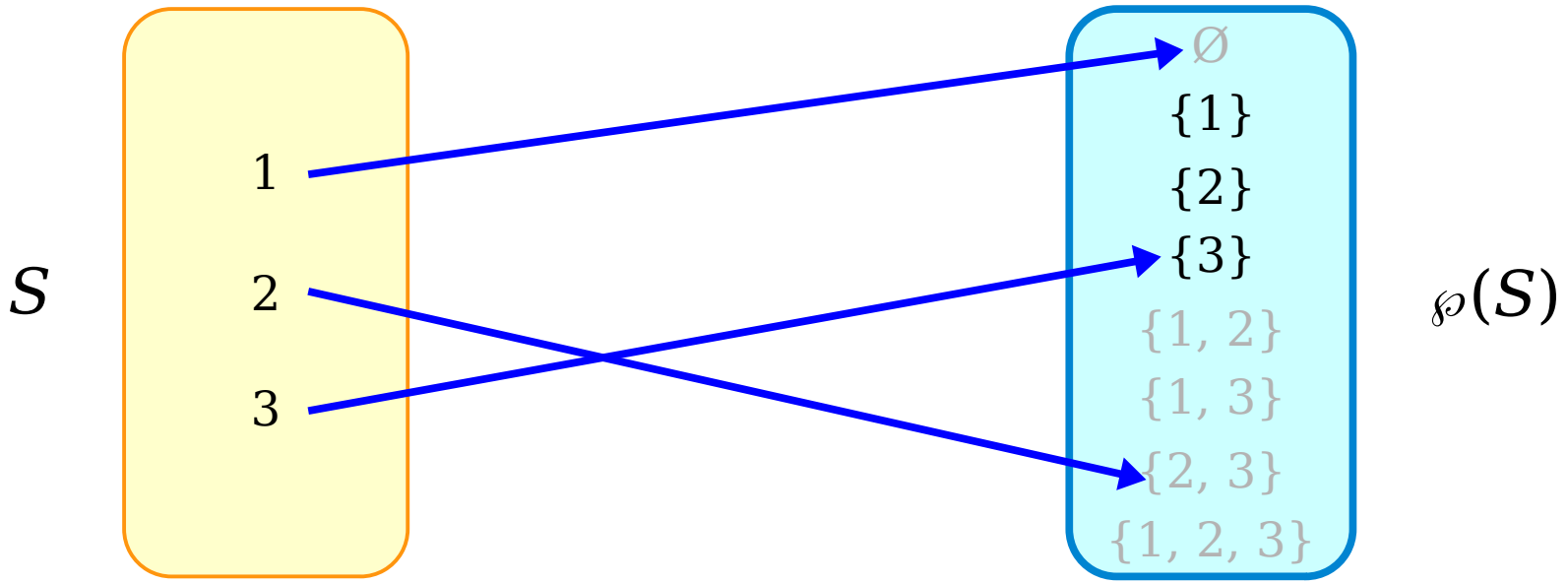
That means that we should try to look for some kind of pattern we can use that will always let us find an injection that works.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



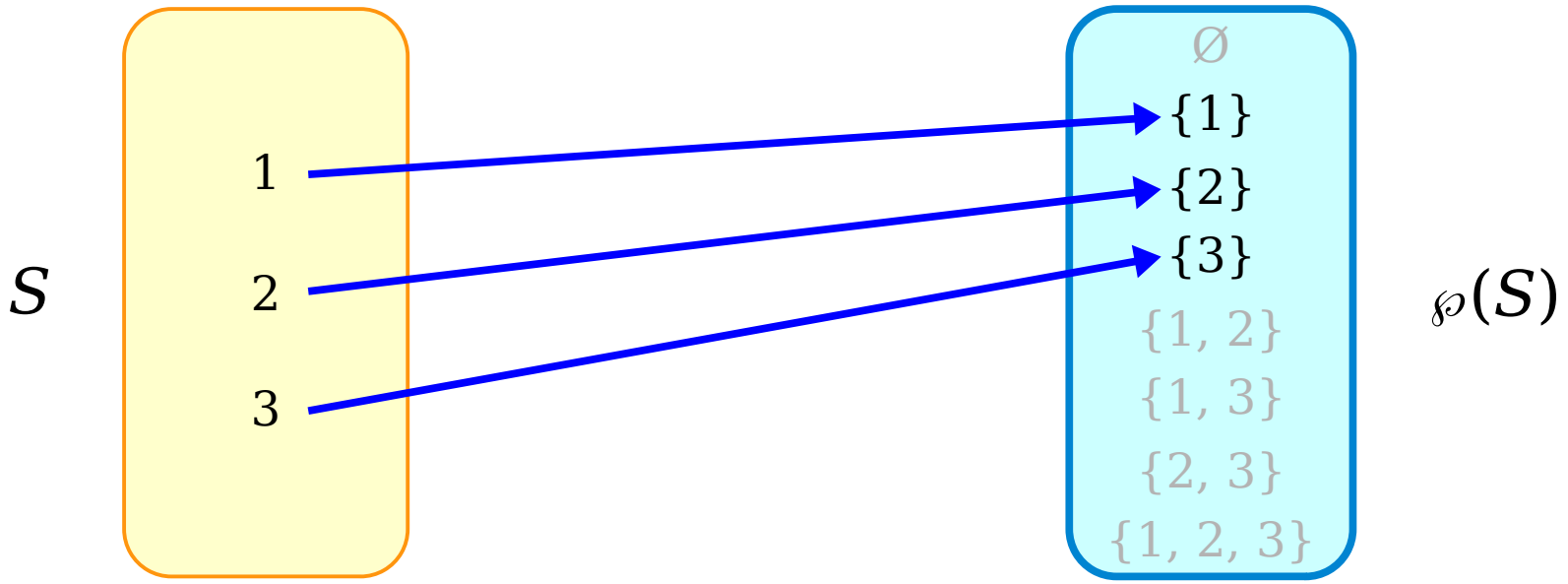
One observation that's useful for doing that here is that for every element of S...

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



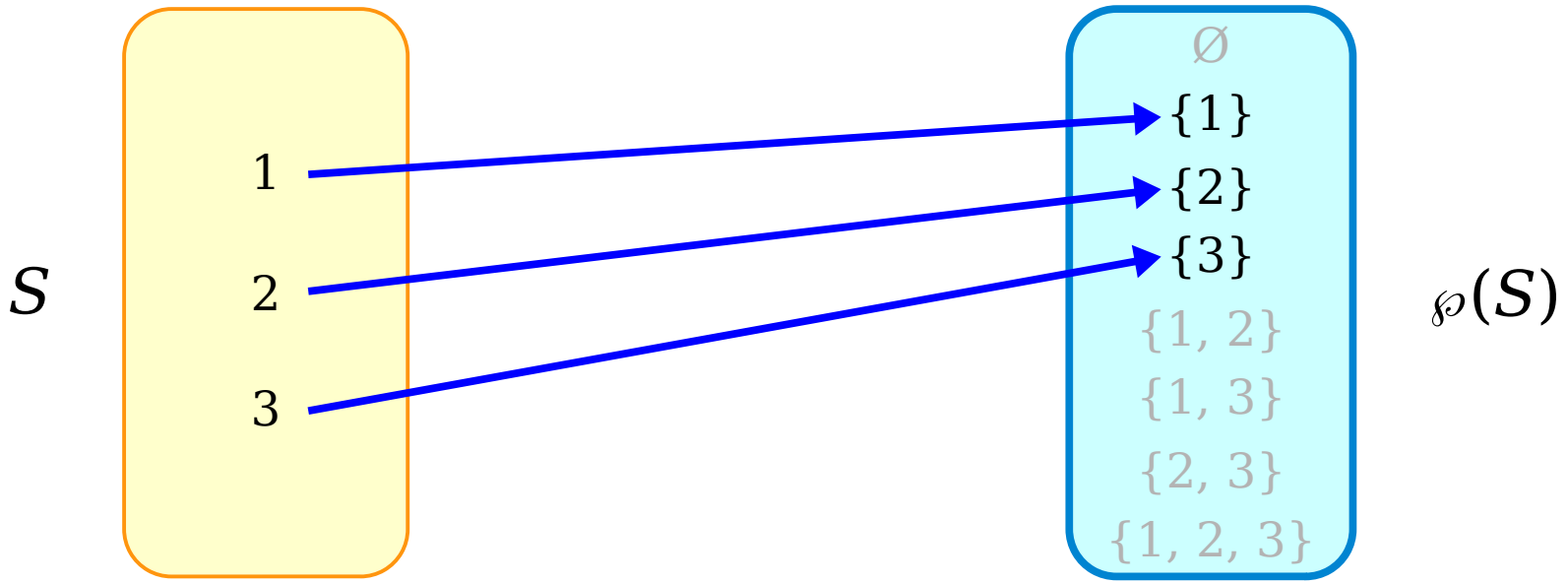
There's a singleton set containing just that element over in $\wp(S)$.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



So perhaps we might want to look at this particular injection.

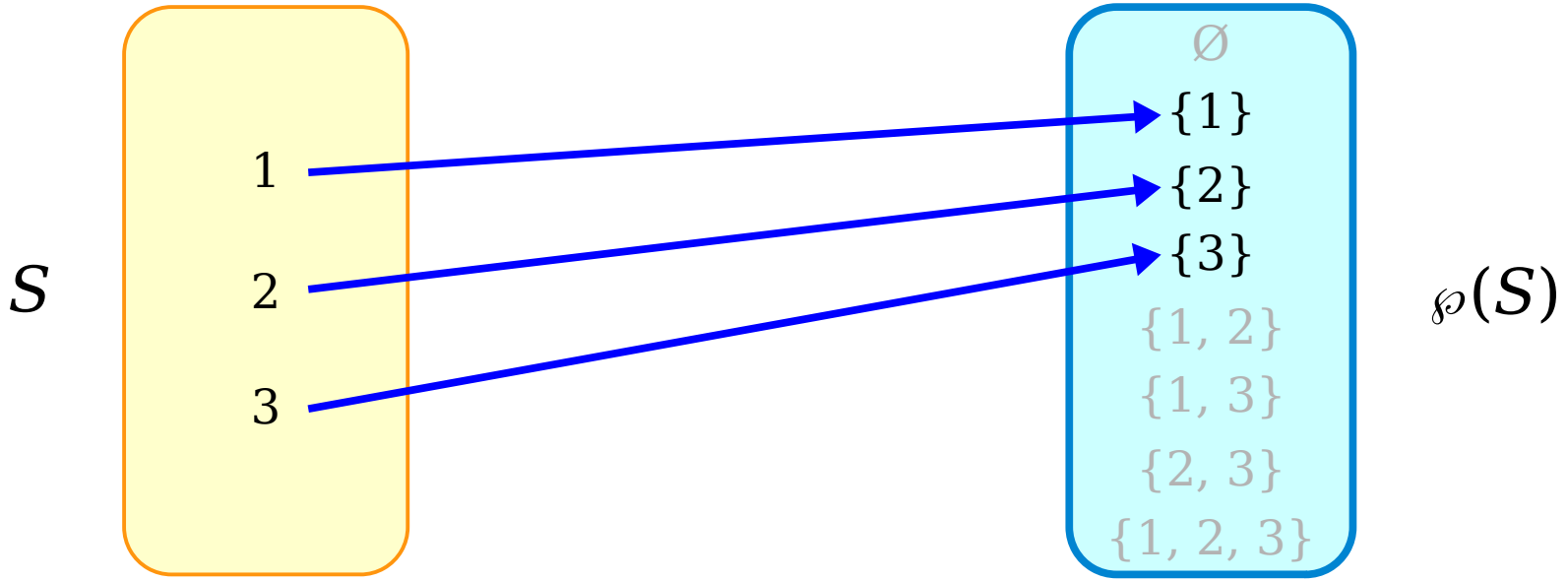
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$

This is a nice injection because there's a clean rule associating elements of S with elements of $\wp(S)$.

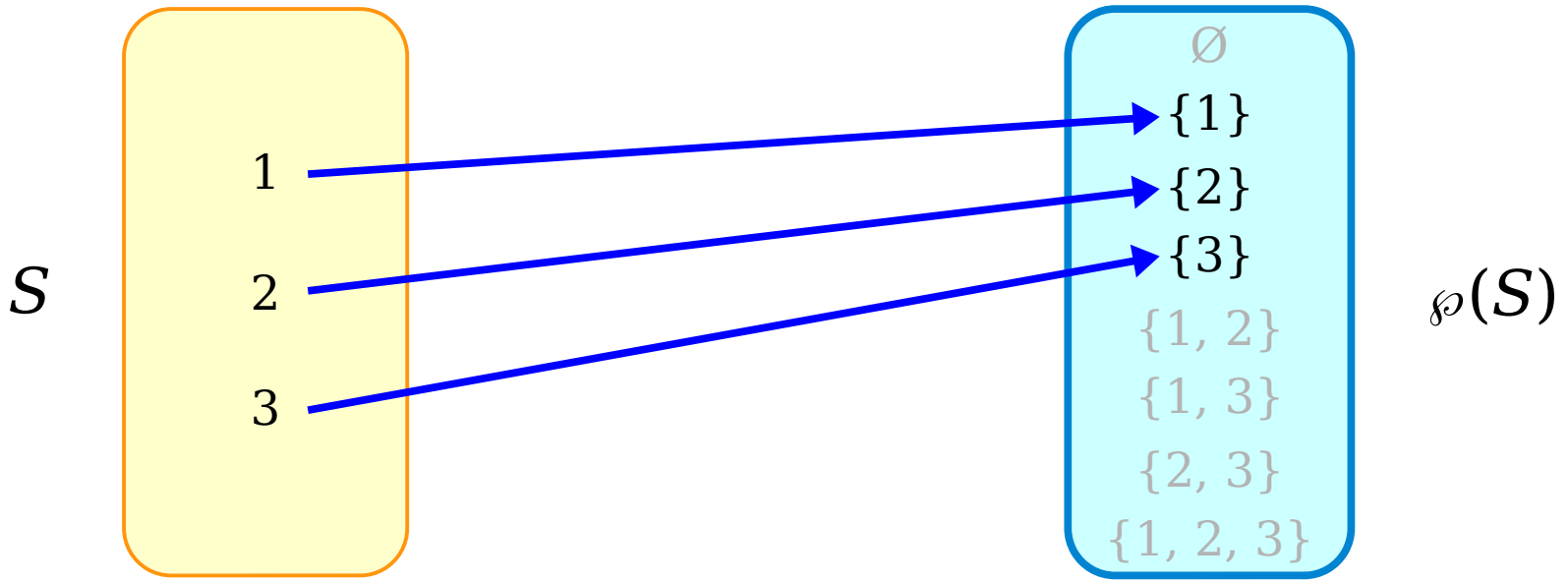
Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$

With a little thought, we can see that no matter what set we choose for S, this function will always be well-defined.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



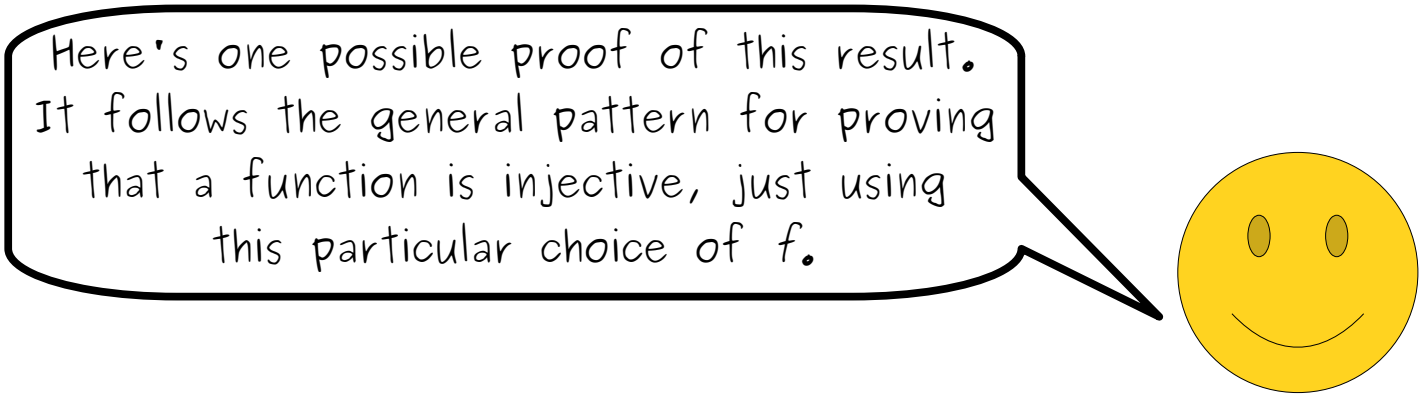
$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$

So if we want to prove the above lemma, perhaps we should try showing that this particular function always works as an injection from S to $\wp(S)$.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.

Proof: Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.

Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required. ■



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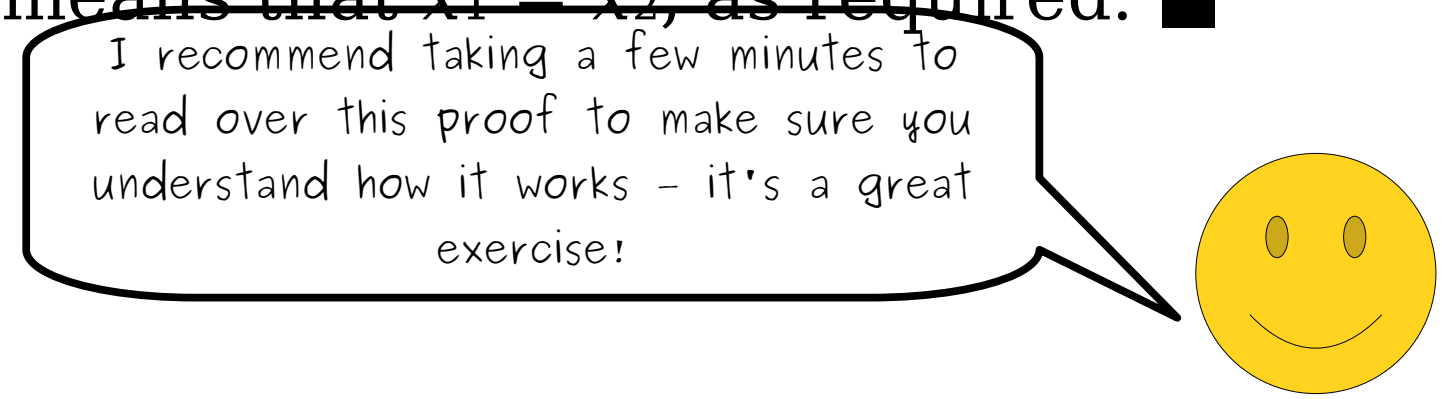
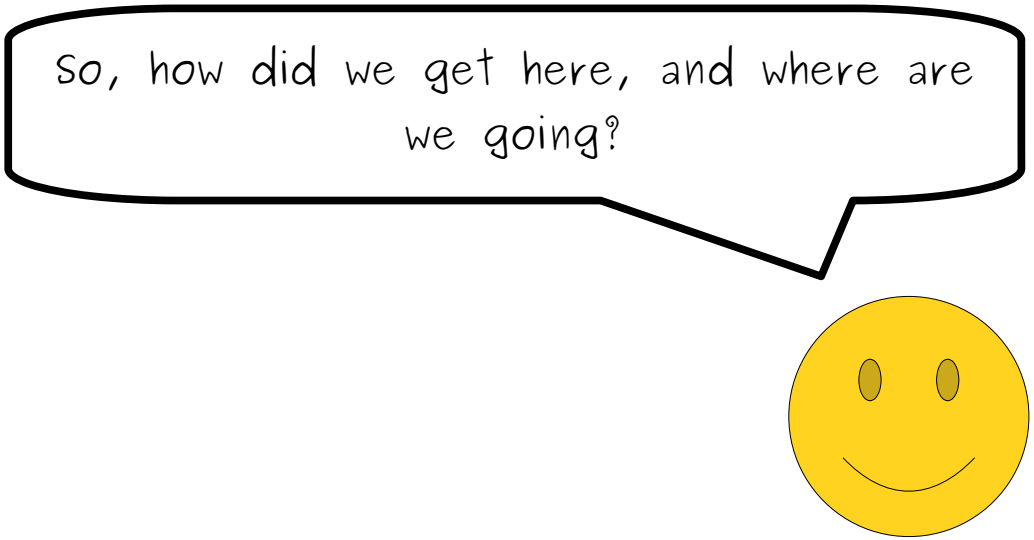
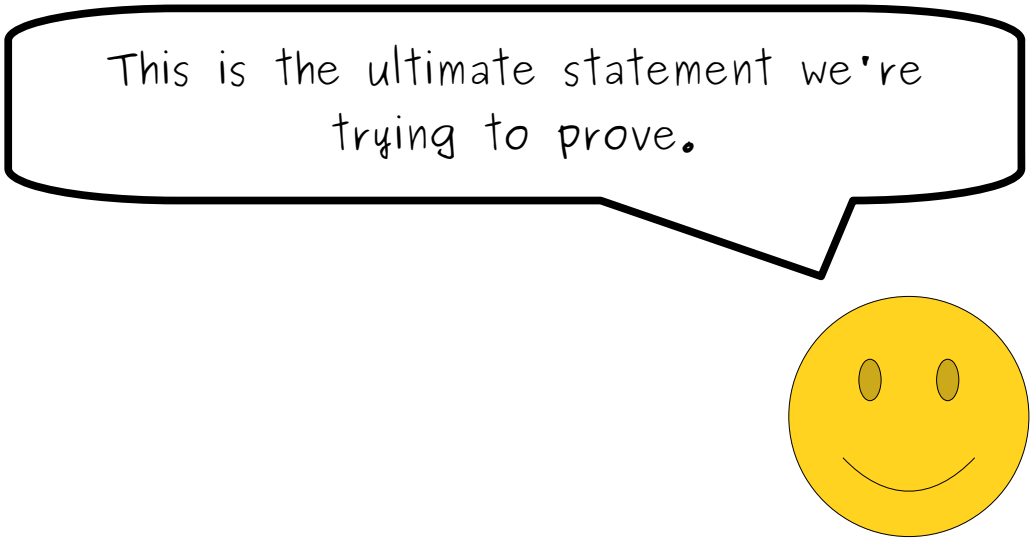



So, how did we get here, and where are we going?

If S is a set, then $|S| < |\wp(S)|$



There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

As we saw earlier, this means that we need to prove these two statements.

There *is* an injection $f : S \to \wp(S)$ ✓
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

We just finished that first step – great!

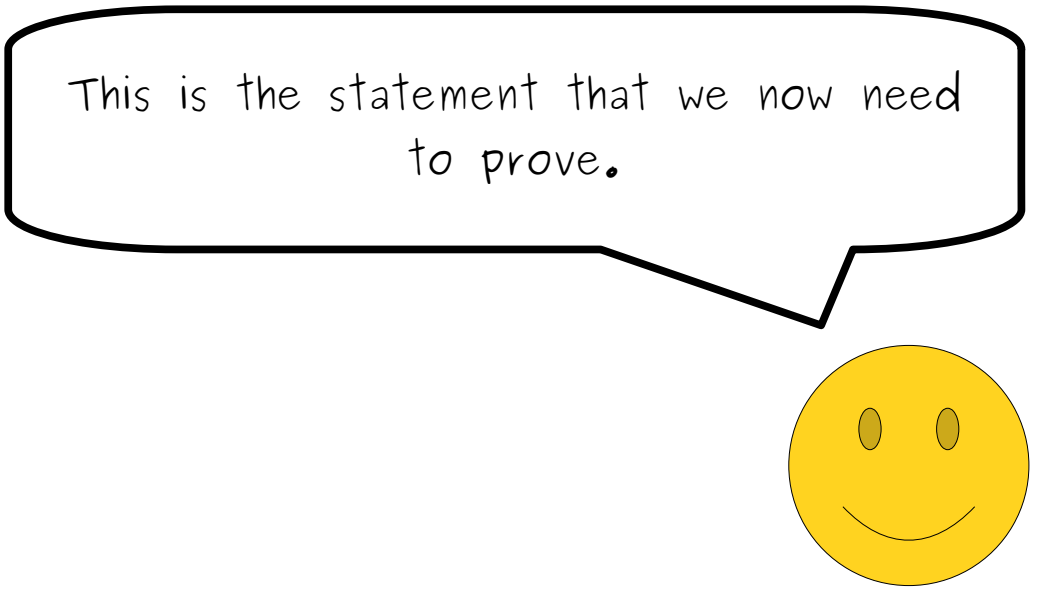
There *is* an injection $f : S \to \wp(S)$ ✓

There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

Now, we need to prove this part… and that's where we're going to use a diagonal argument.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



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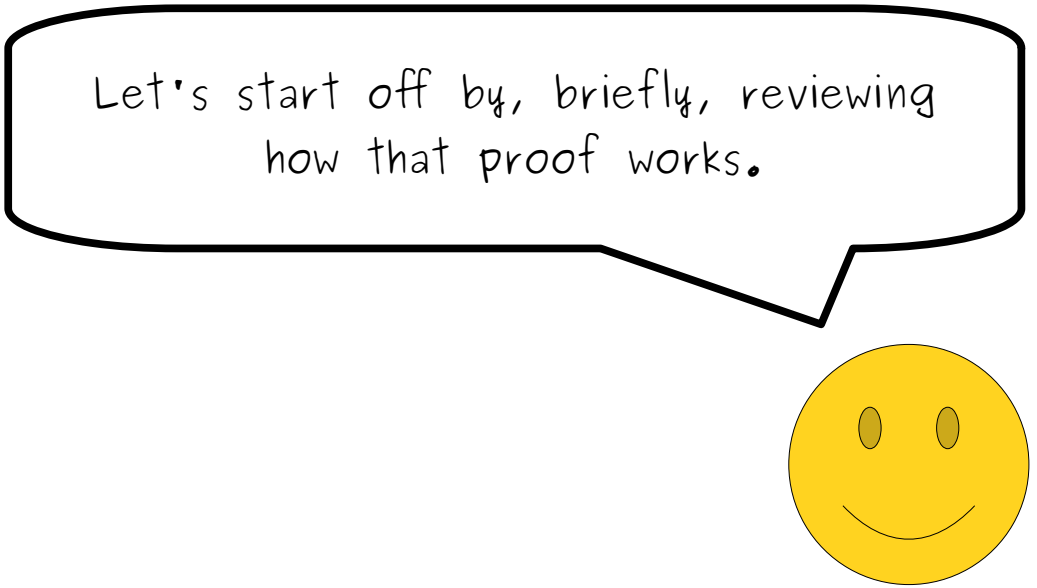


Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0

x_1

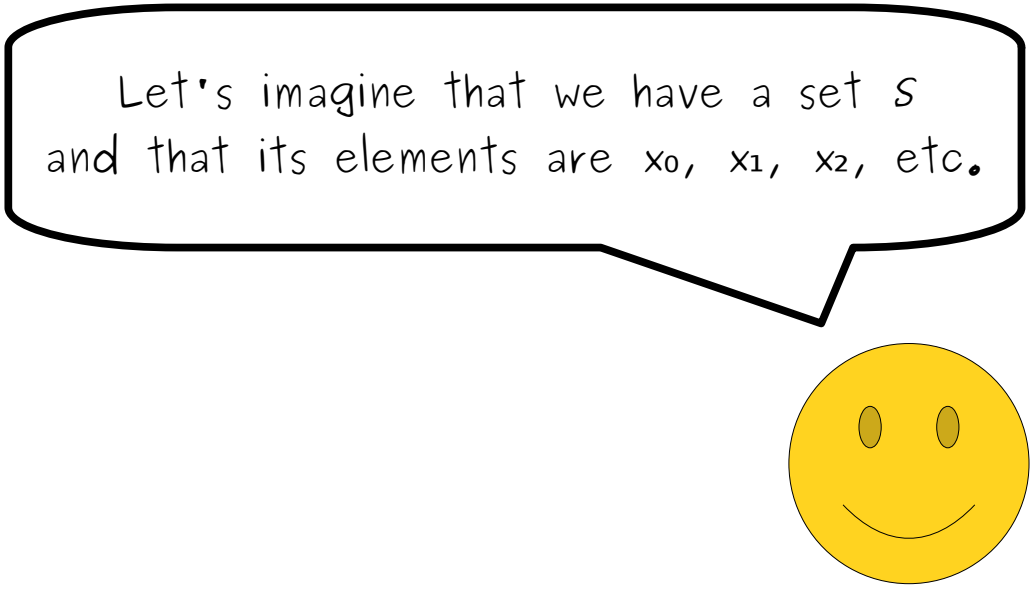
x_2

x_3

x_4

x_5

...



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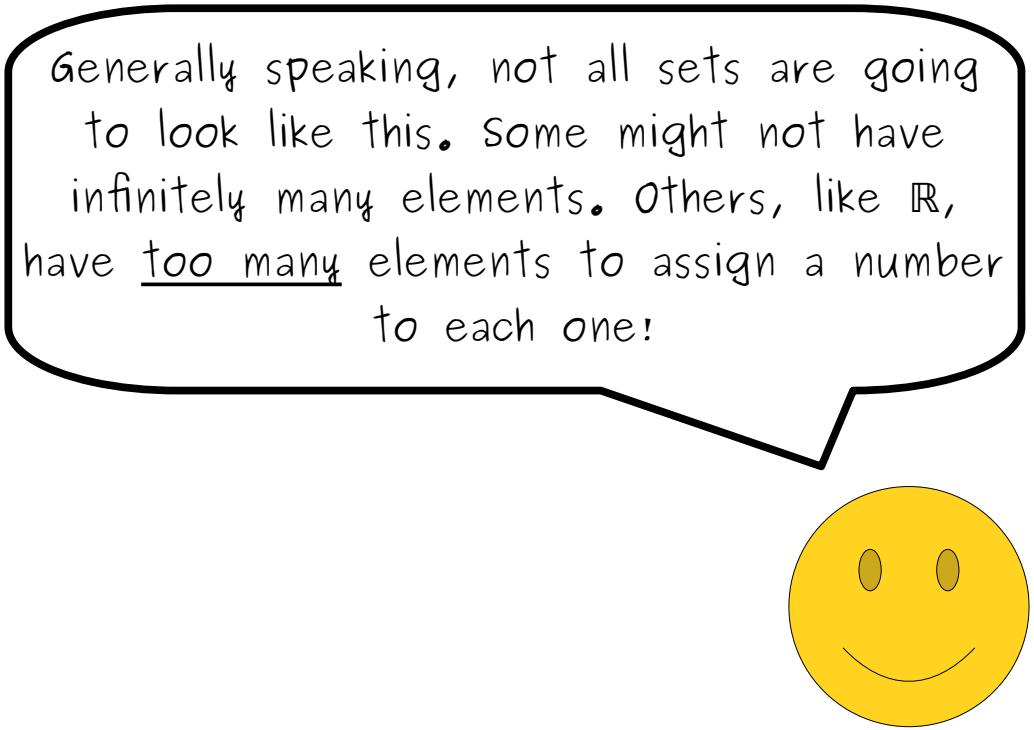

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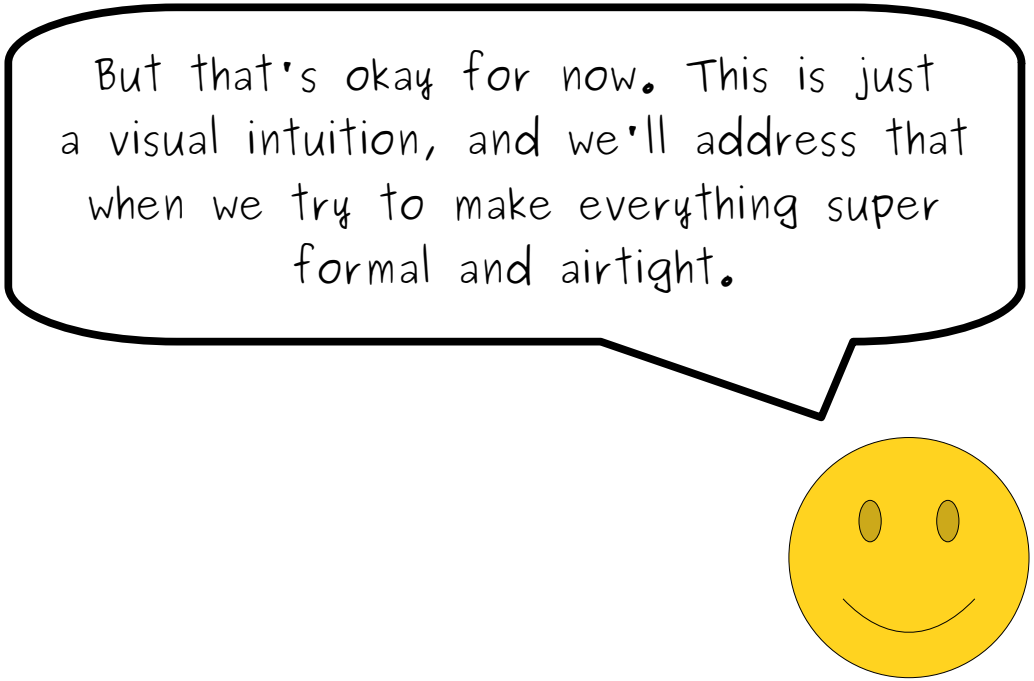

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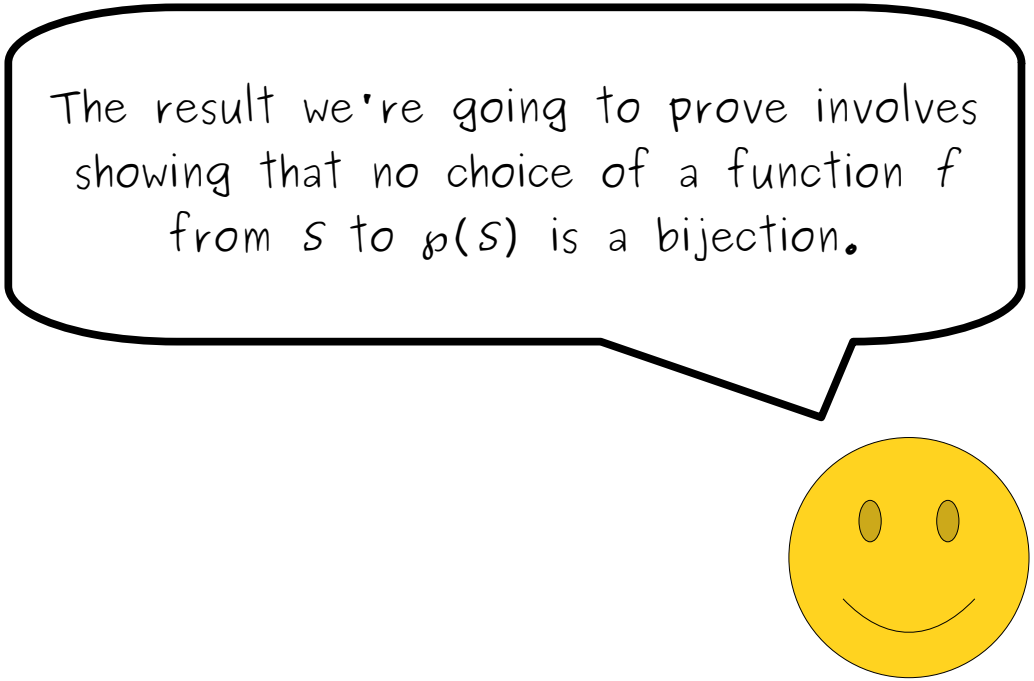

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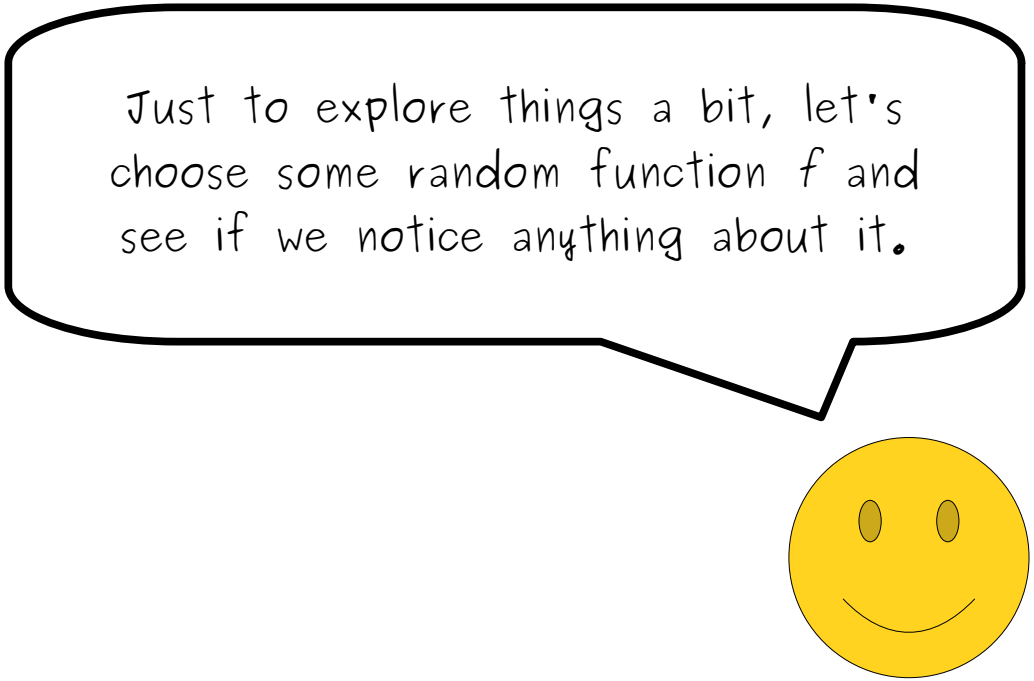
...



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$x_0 \longrightarrow \{ x_0, x_2, x_4, \ldots \}$

$x_1 \longrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \longrightarrow \{ x_4, \ldots \}$

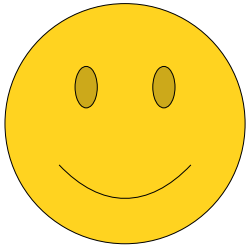
$x_3 \longrightarrow \{ x_1, x_3, x_4, \ldots \}$

$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$

Here's one example function f. There's no particular pattern here in the function we picked – it's just a random function to help us build an intuition.



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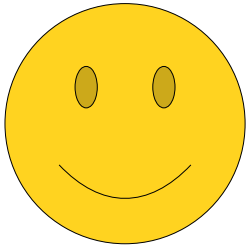
$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$

Our ultimate goal is to prove that this function is not bijective.



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$x_0 \longrightarrow \{ x_0, x_2, x_4, \ldots \}$

$x_1 \longrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \longrightarrow \{ x_4, \ldots \}$

$x_3 \longrightarrow \{ x_1, x_3, x_4, \ldots \}$

$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

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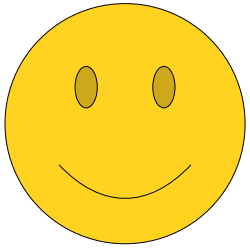
$\ldots$

That's going to be a challenge, since we can't assume that this function follows any nice pattern.



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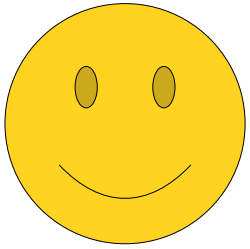
$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$

Our approach will be to show that this function is not surjective. If we can do that, then we can say with certainty that f is not bijective.



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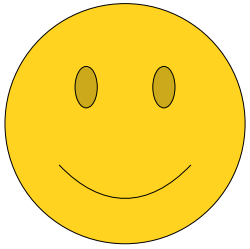
$\ldots$

To do so, we're going to have to find a way to show that there's some subset of S that isn't covered by our function.



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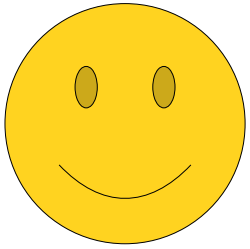
$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$

Visually, that means that we need to find a set that's not paired with anything in the left-hand column.



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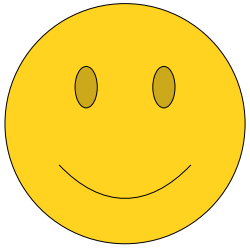
$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

$\ldots$

To do so, we're going to use a trick invented by Georg Cantor, which will require us to redraw this picture a bit.



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x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, x_2, x_4, \ldots \}$

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$x_3 \longrightarrow \{ x_1, x_3, x_4, \ldots \}$

$x_4 \longrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

We're going to start off by writing the elements of S across the top, like this.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

…

We can then imagine spacing out the elements that are in each of these sets so that we start to have something that looks more like a grid.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

…

Cantor's insight – which is the real trick behind the proof – is the following.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \to \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \to \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

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$x_3 \to \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

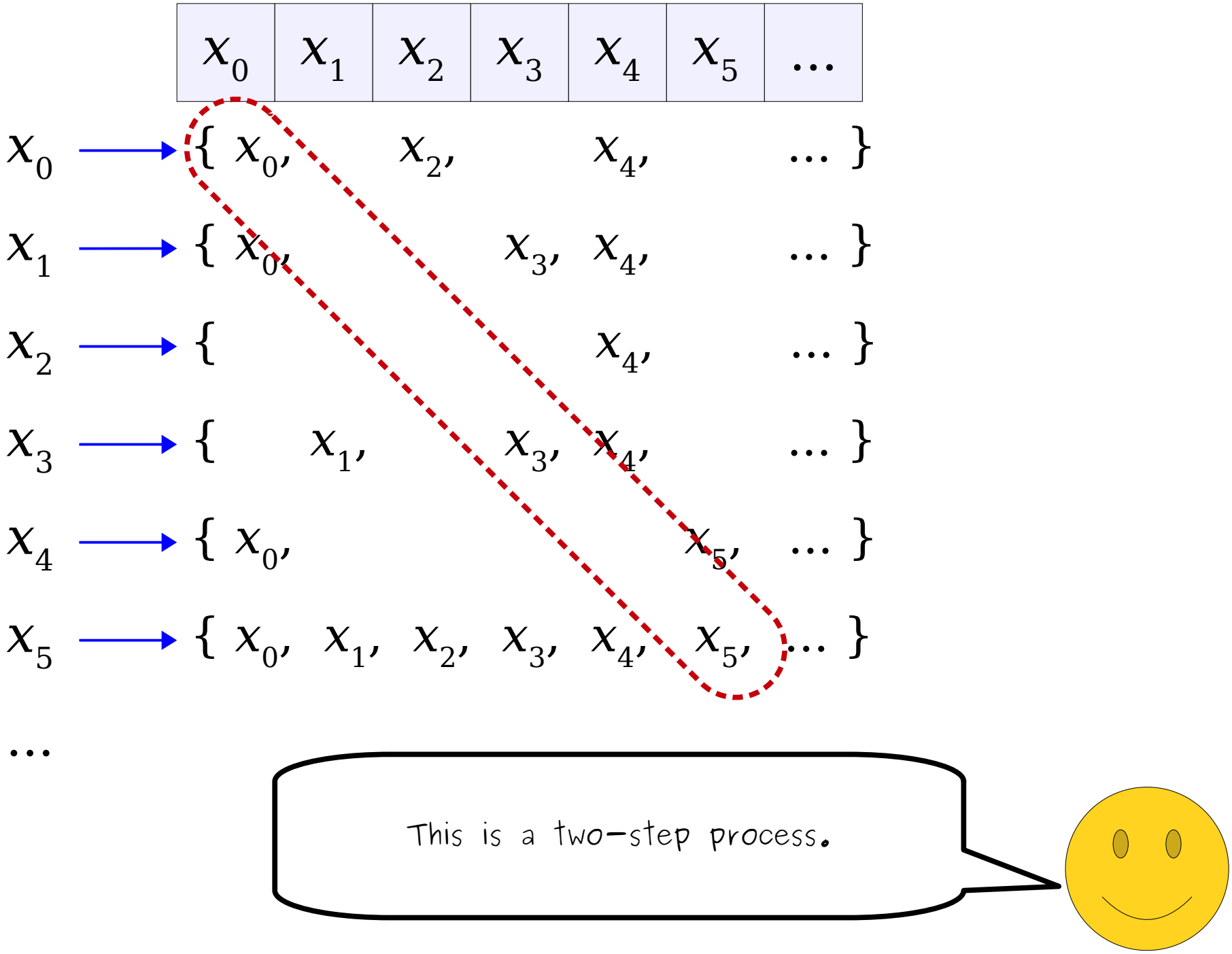
$x_4 \to \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \to \{ x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \}$

…

The trick, as you saw in lecture one, is to look at this main diagonal and use it to build a set that's not mapped to.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



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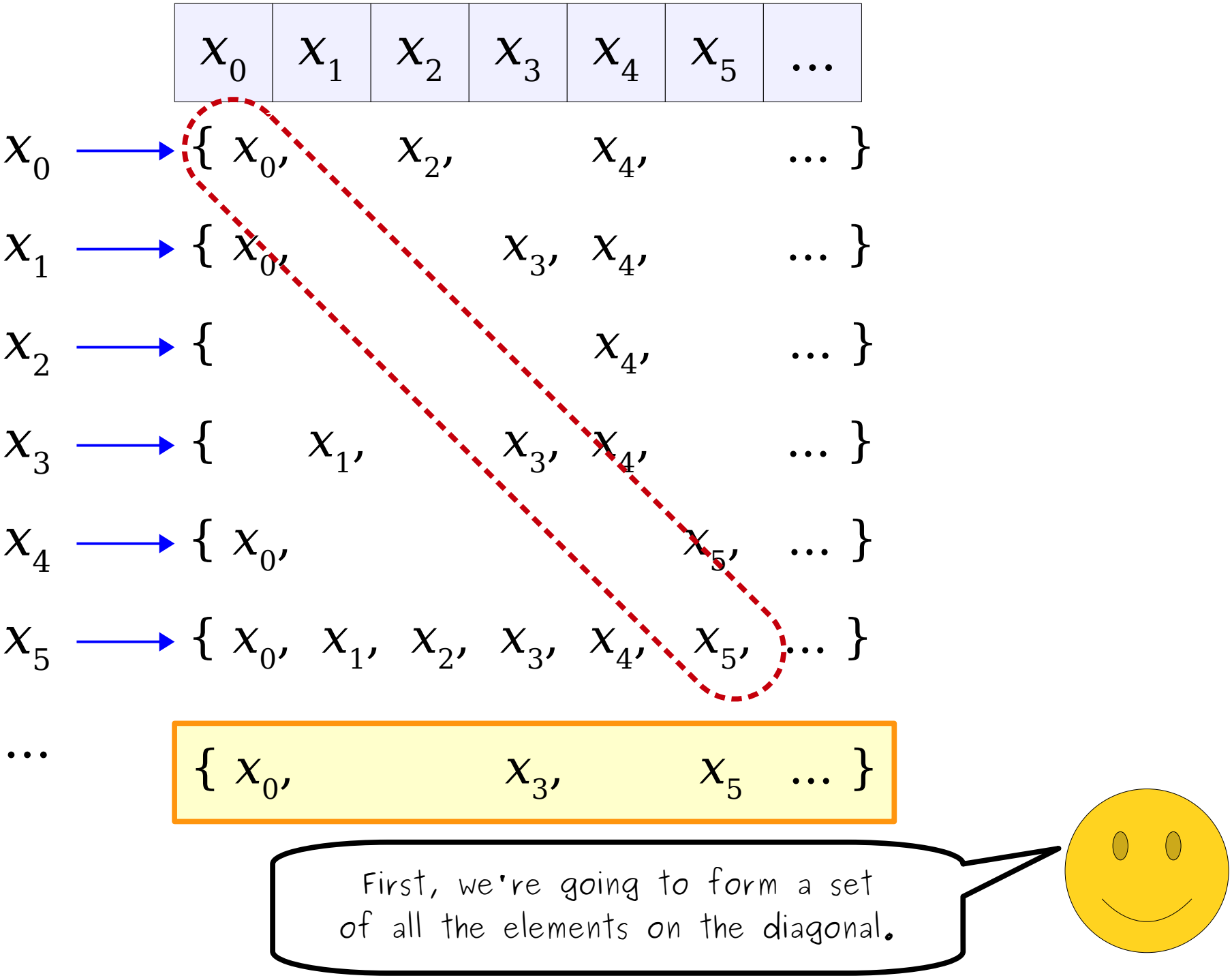

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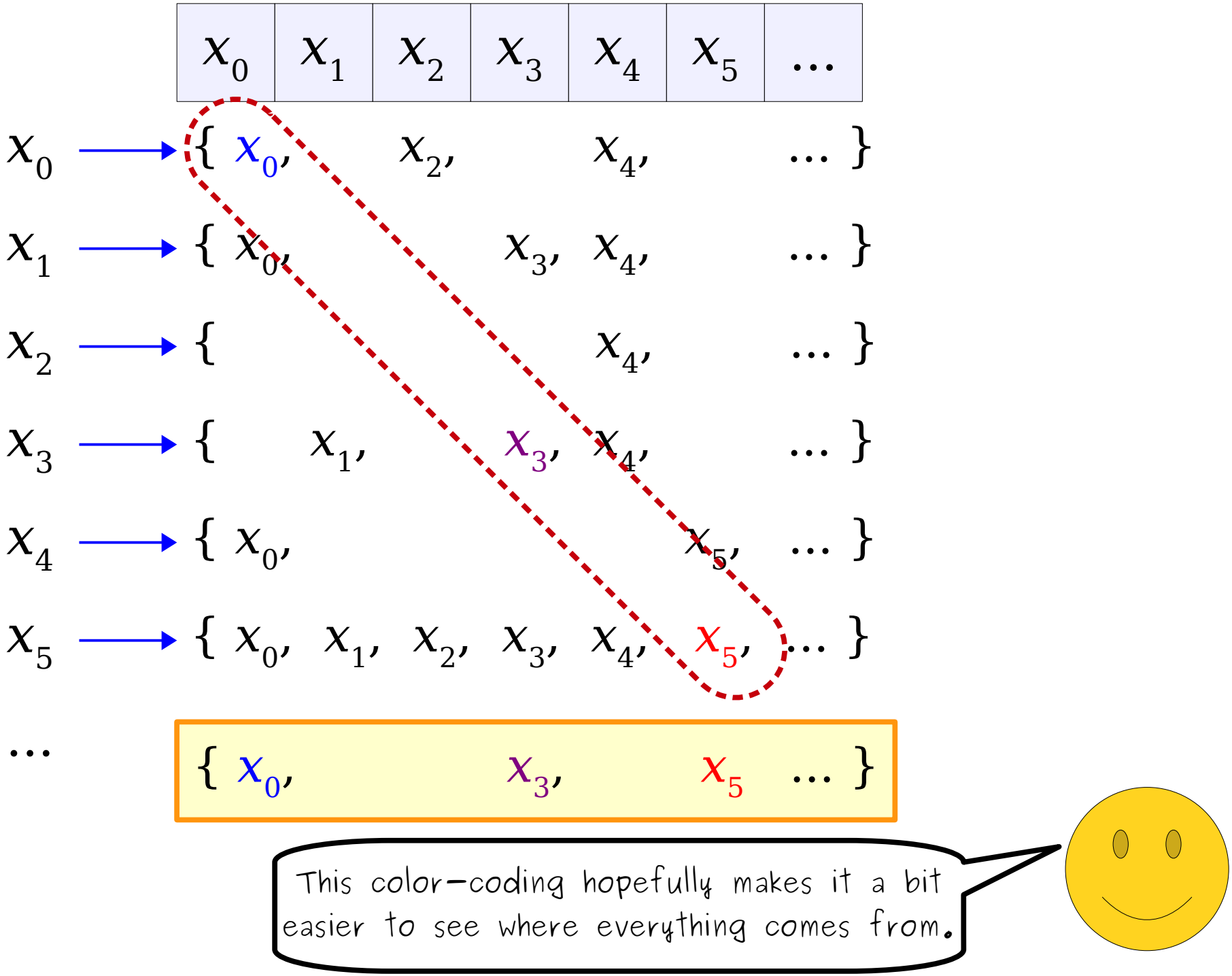

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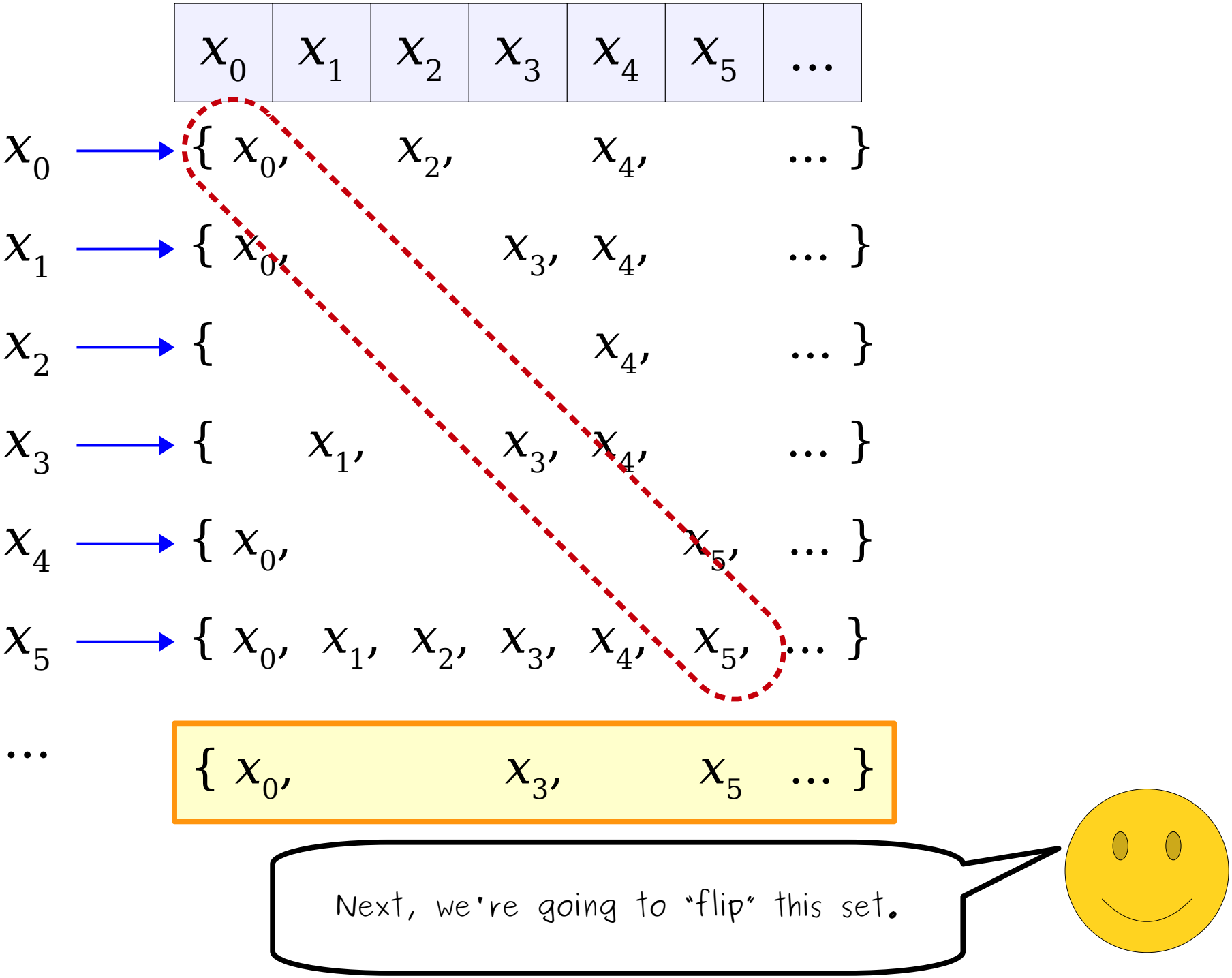

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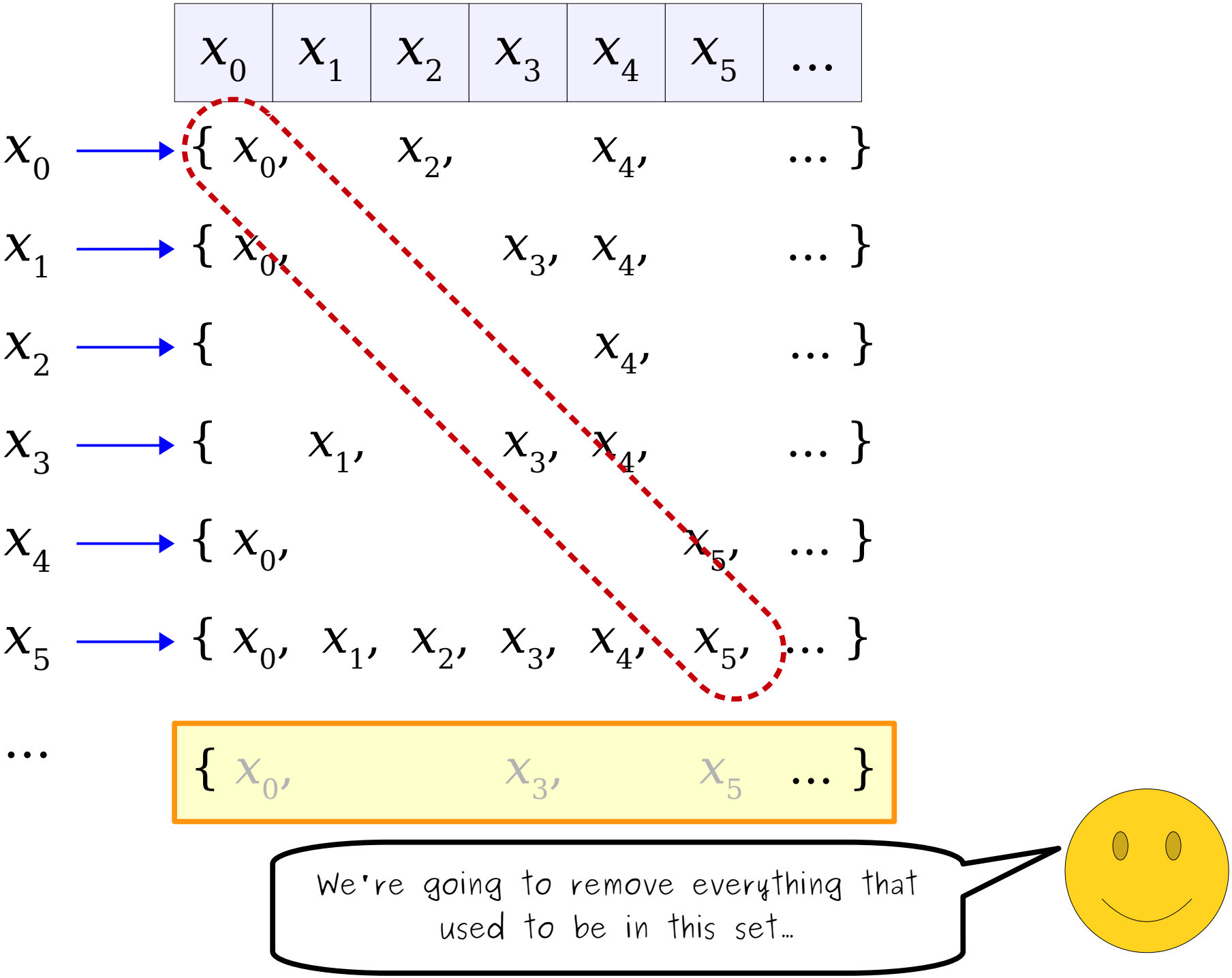

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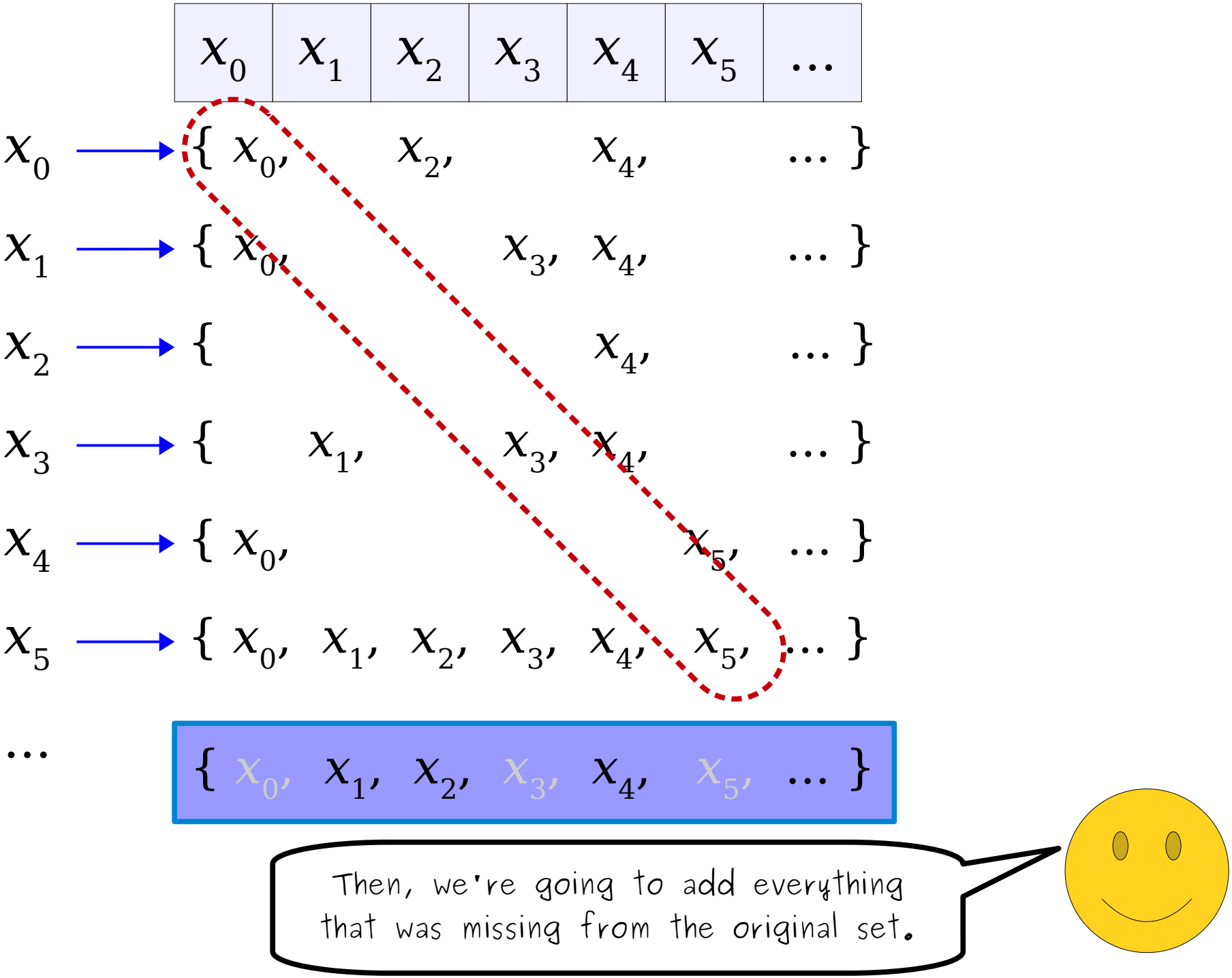

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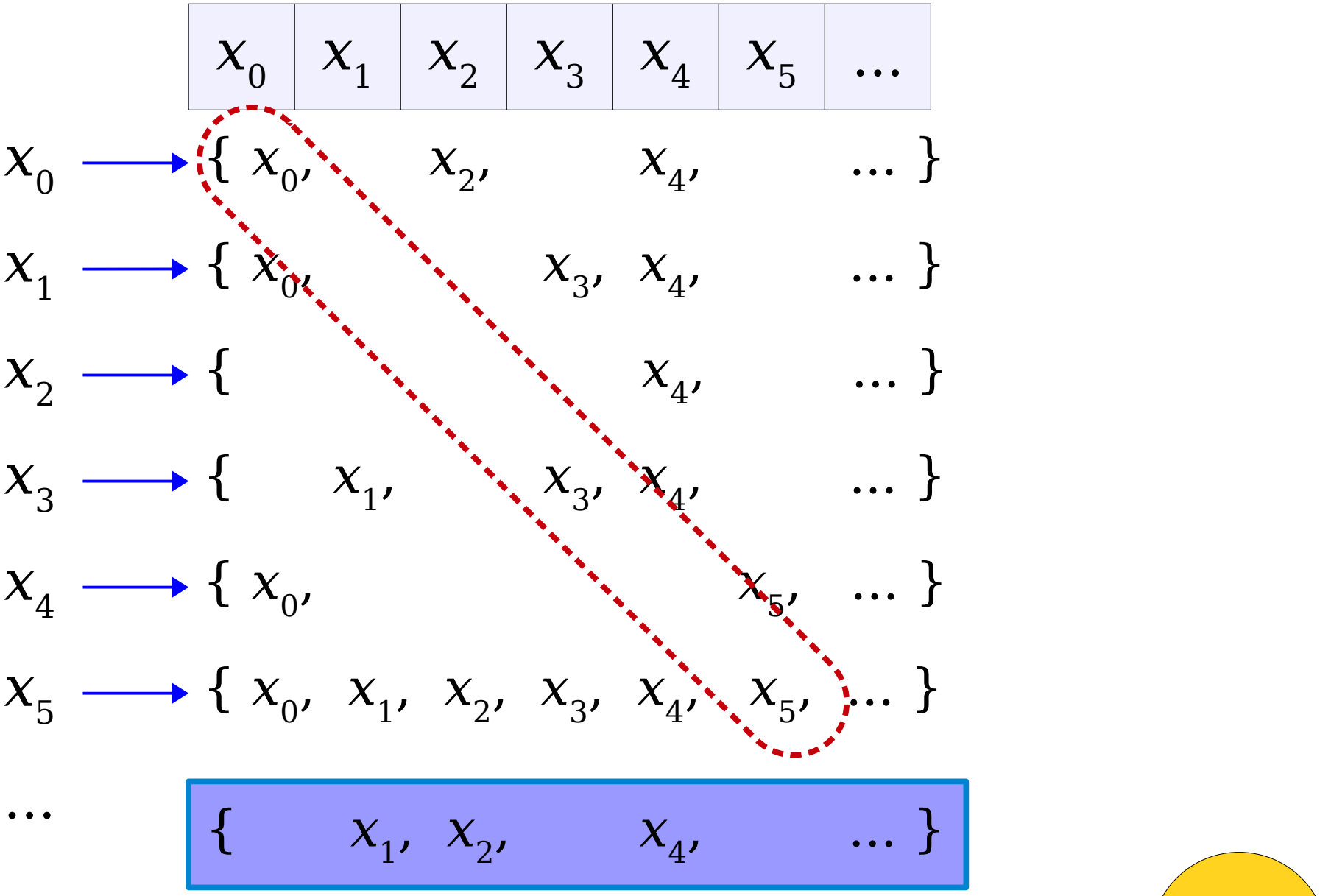

This new set – which is called the diagonal set – is specifically built to be different from every row above.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

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$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$\ldots$ $\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

Let's see why this is.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \to \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

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$x_2 \to \{ \quad x_4, \quad \dots \}$

$x_3 \to \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \to \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \to \{ x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \}$

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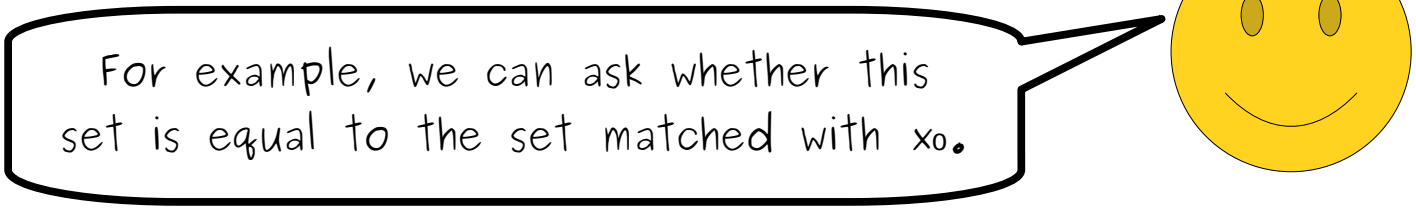

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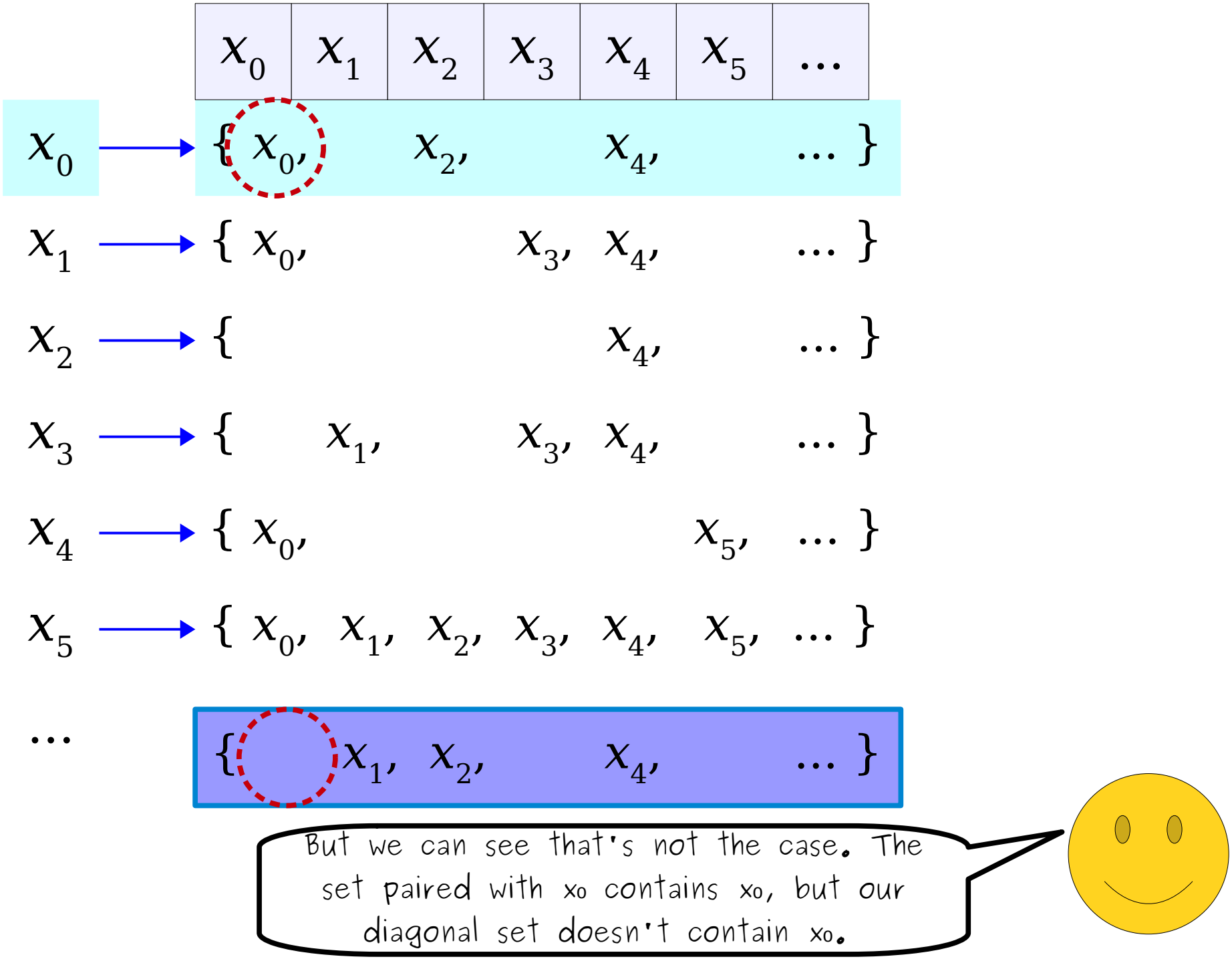

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$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

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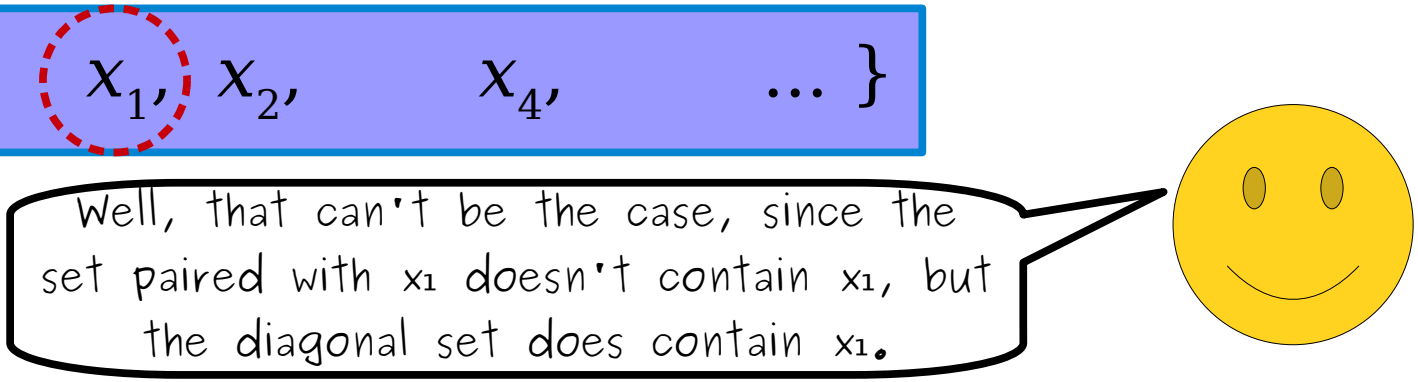

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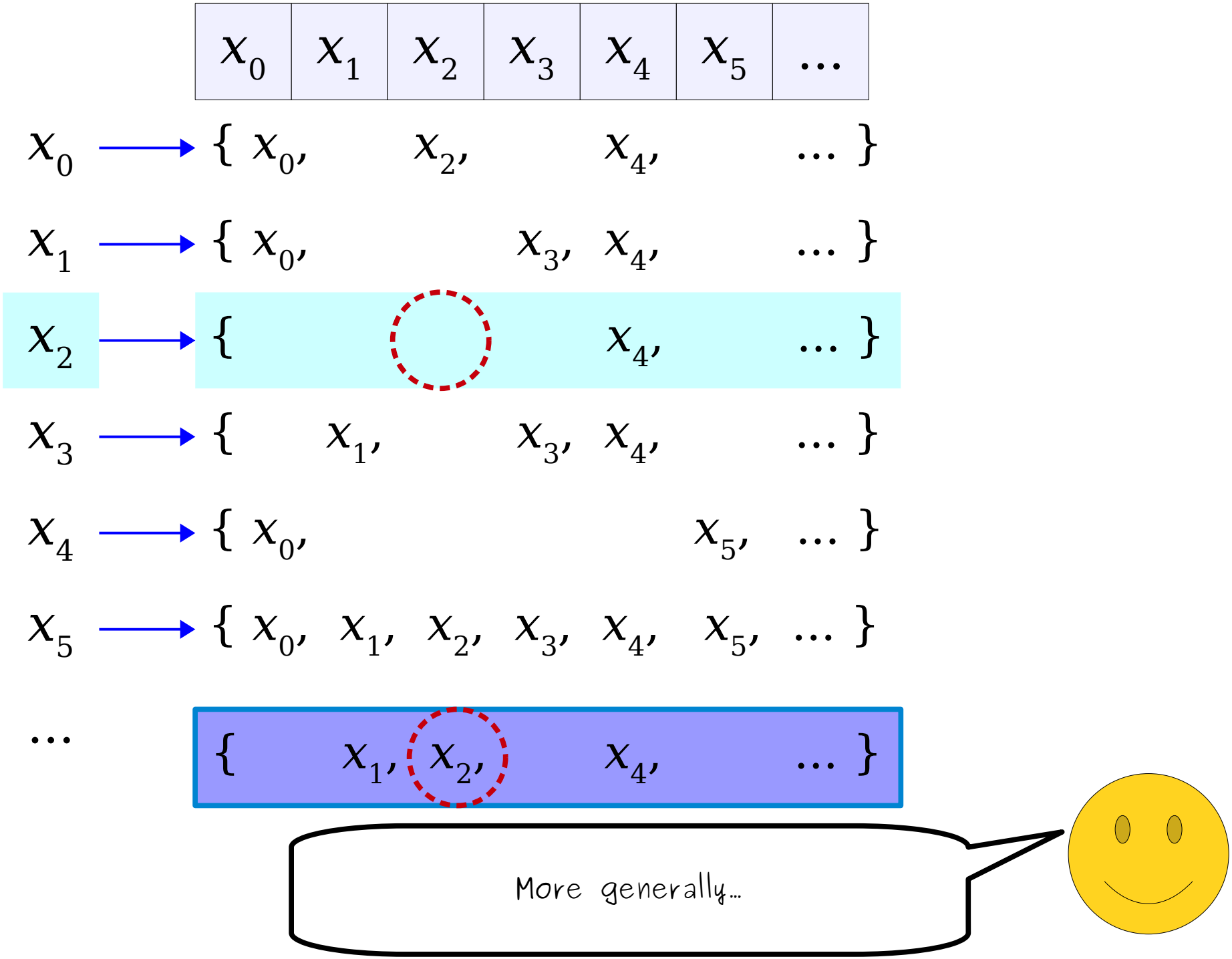

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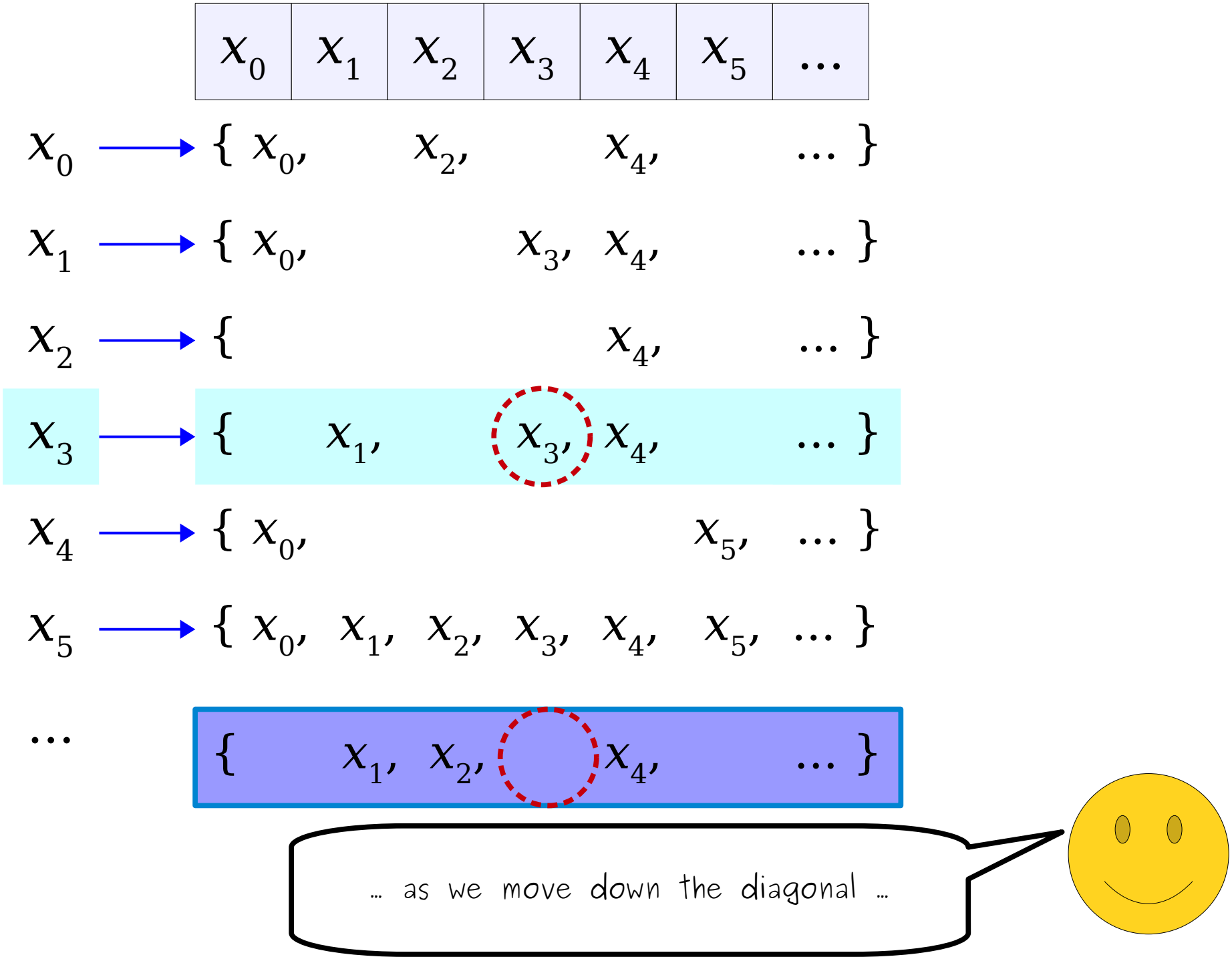

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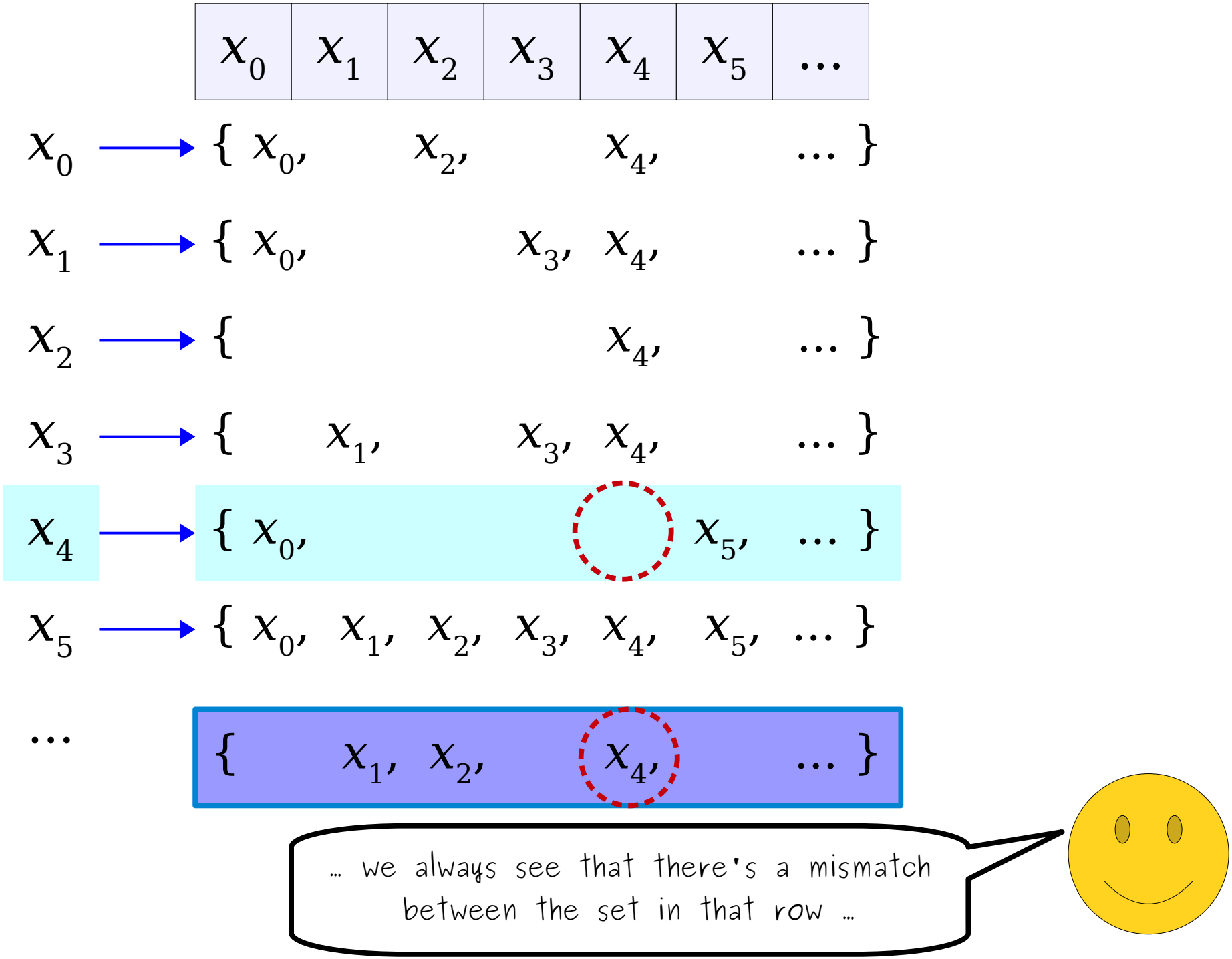

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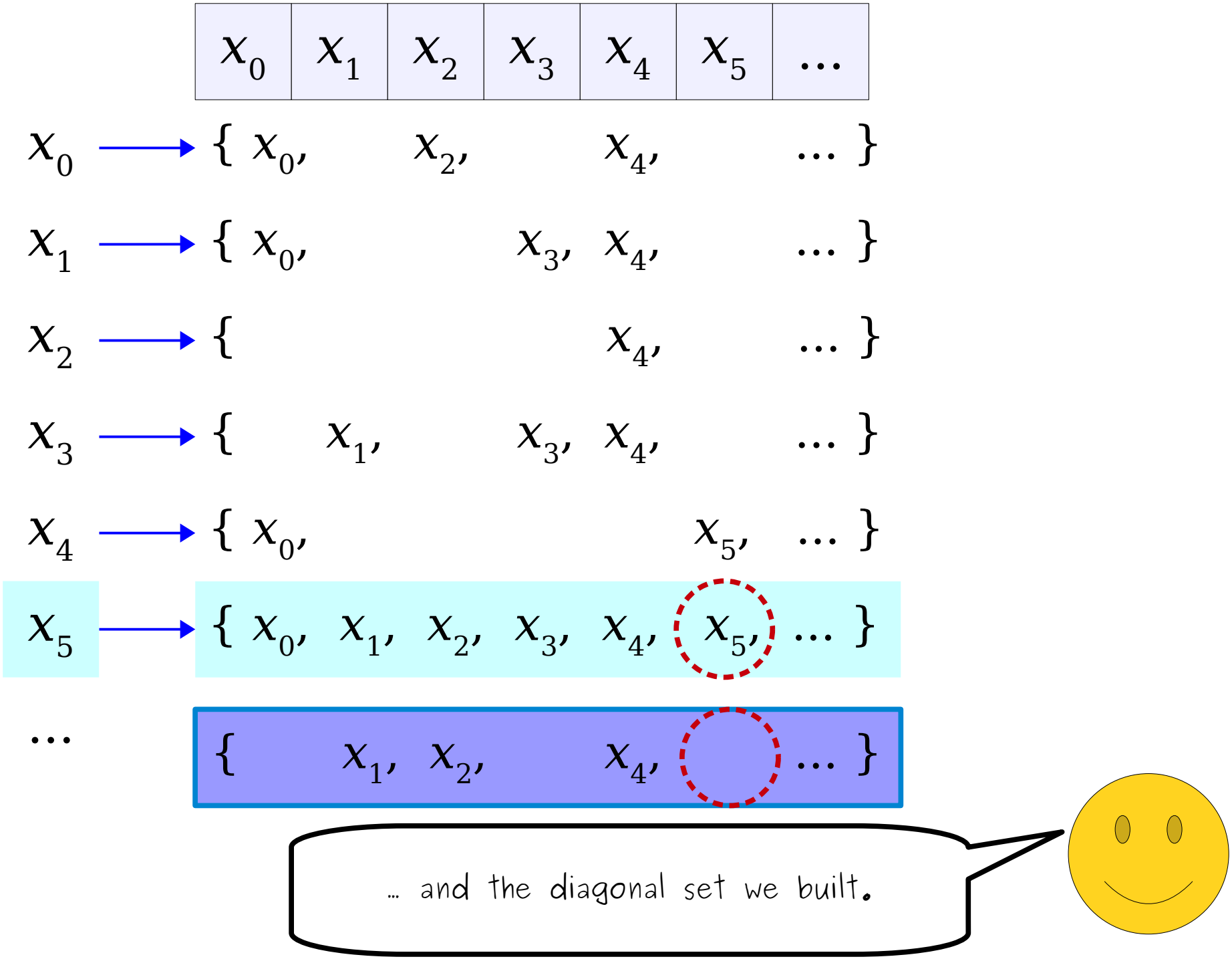

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$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \}$

$\dots \quad \{ \quad x_1, \; x_2, \quad x_4, \quad \dots \}$

As a result, we know that this new set isn't paired with anything.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

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$x_5 \longrightarrow \{ x_0, \; x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; \dots \}$

$\dots$ $\{ \quad x_1, \; x_2, \quad x_4, \quad \dots \}$

Set D

Formally speaking, if we call our set D…

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$f : S \to \wp(S)$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

… and this particular function is f…

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$f : S \to \wp(S)$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Then we see that $f(x) \neq D$ for any $x \in S$, meaning that f isn't surjective!

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\ x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

$x_3 \longrightarrow \{\quad x_1,\quad x_3,\ x_4,\quad \ldots\ \}$

$x_4 \longrightarrow \{\ x_0,\quad x_5,\quad \ldots\ \}$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$f : S \to \wp(S)$

…

$\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

Set D

The beauty behind this construction is that it doesn't matter what choice of f we make.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

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$x_4 \longrightarrow \{\ x_0,\quad x_5,\quad \ldots\ \}$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$f : S \to \wp(S)$

…

$\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

Set D

If we take the diagonal and flip it, we'll always find a set that nothing maps to.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

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$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$f : S \to \wp(S)$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Our challenge now is to convert this picture into a rigorous proof.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$f : S \to \wp(S)$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

To do that, we're going to need to find a way to rigorously define the set D.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$f : S \to \wp(S)$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Once we've done that, we can write a formal proof that $f(x) \neq D$ for any $x \in S$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$

$\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Since sets are uniquely defined by the elements they contain, a reasonable first step is to see what's in D.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

$x_3 \longrightarrow \{\quad x_1,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_4 \longrightarrow \{\ x_0,\quad x_5,\quad \ldots\ \}$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$\ldots$

$\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

Set D

Although in this discussion I'm going to use the example f given here, this will work for any choice of f.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

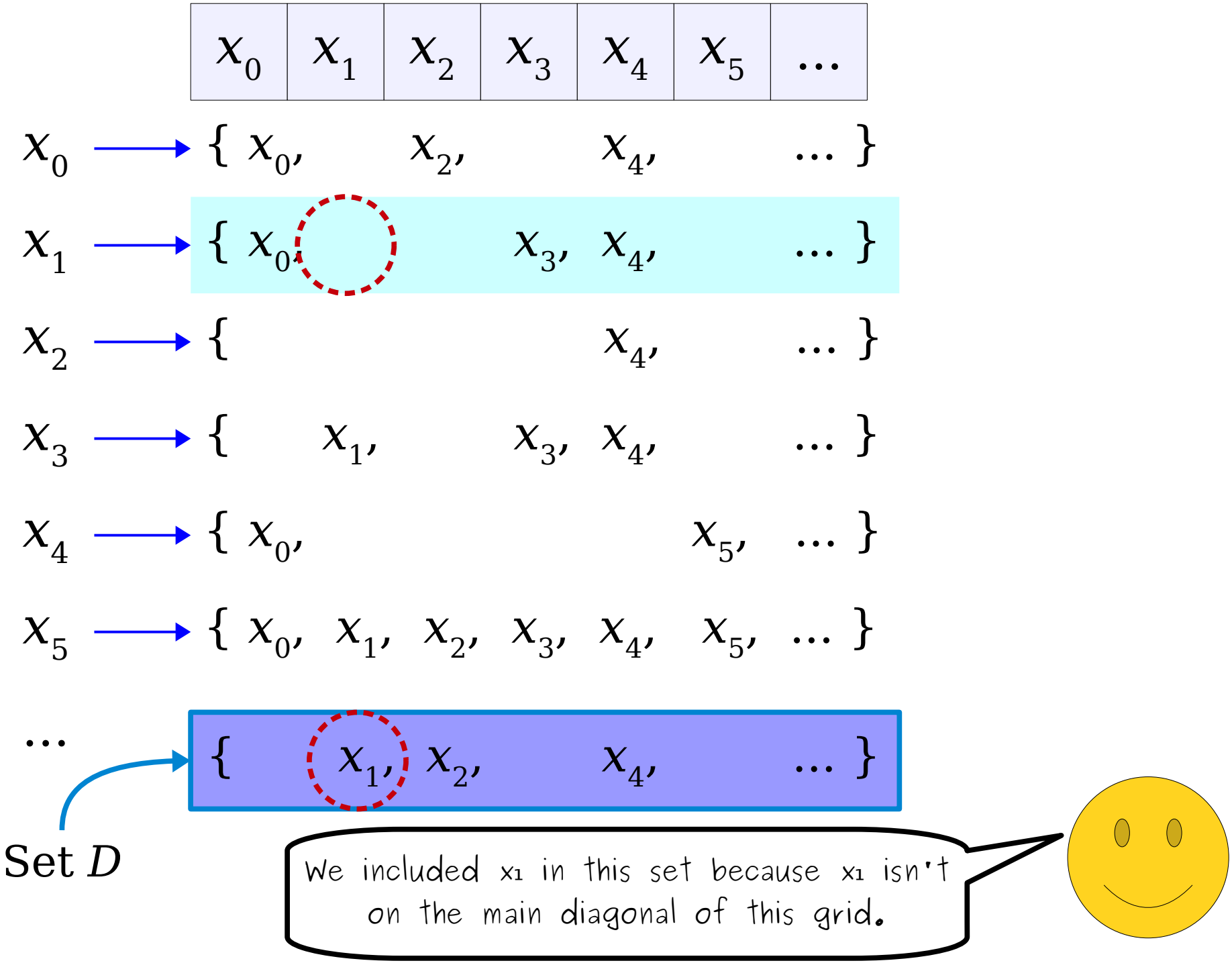
$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

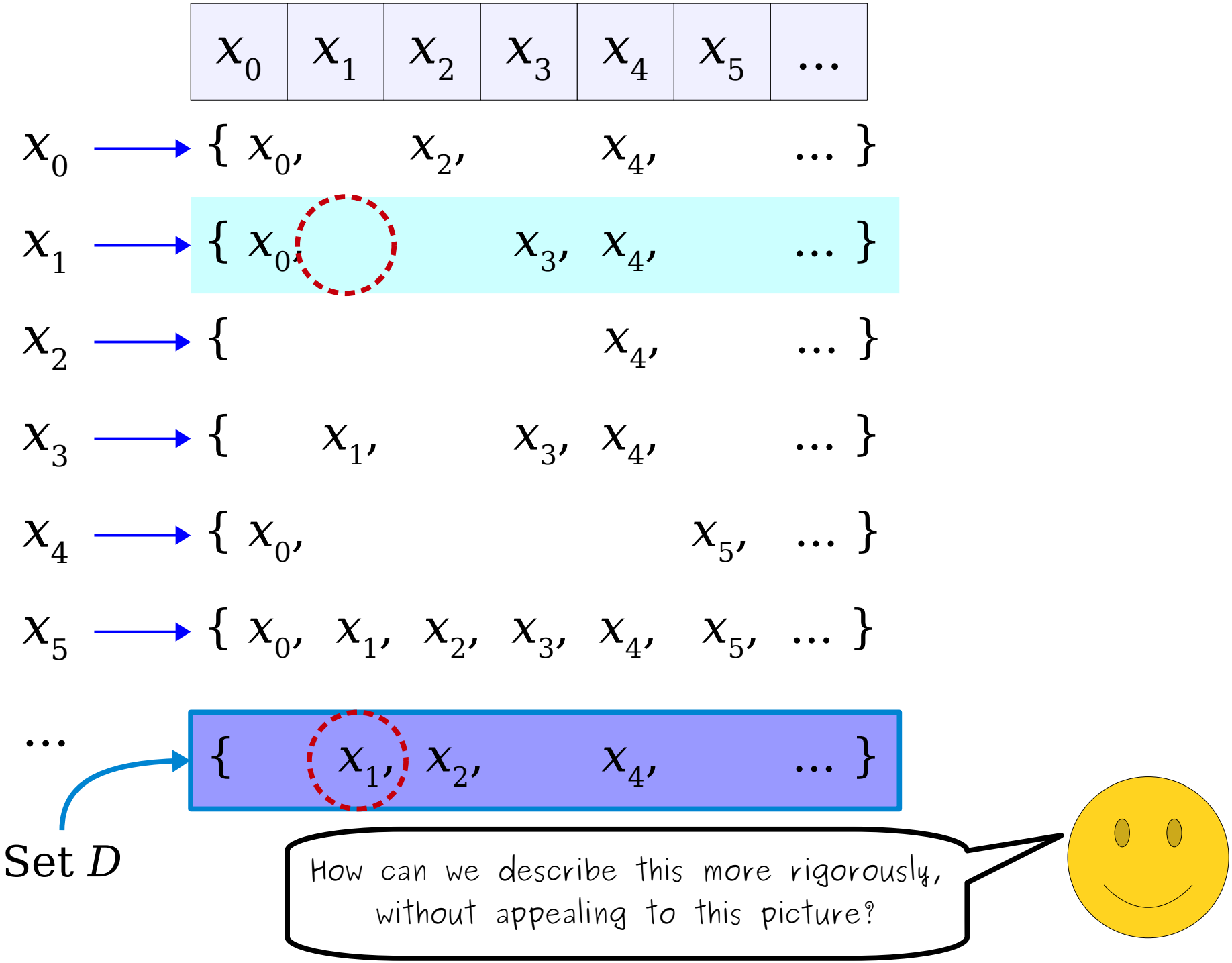
Set D

Let's look at x_1, for example.

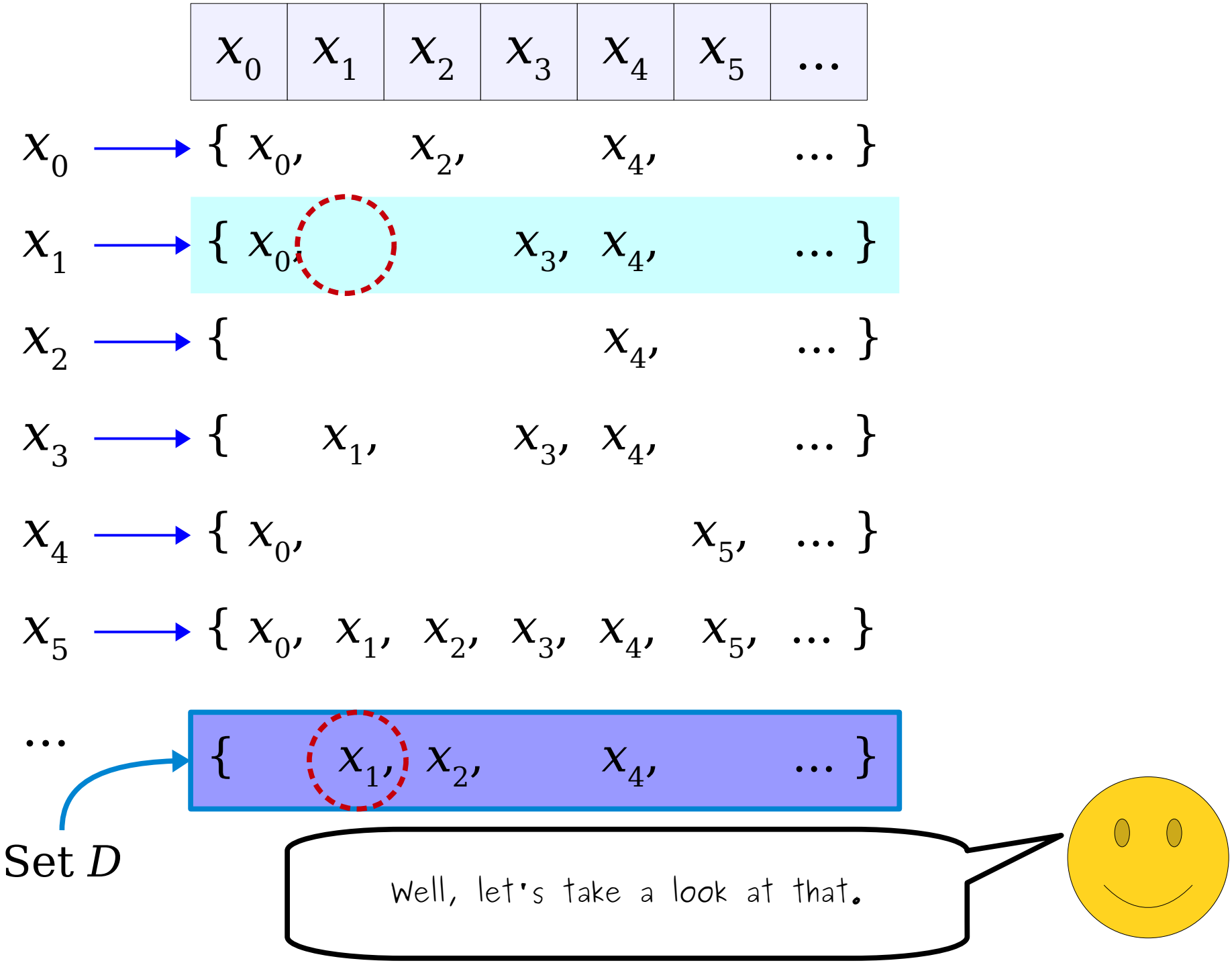
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



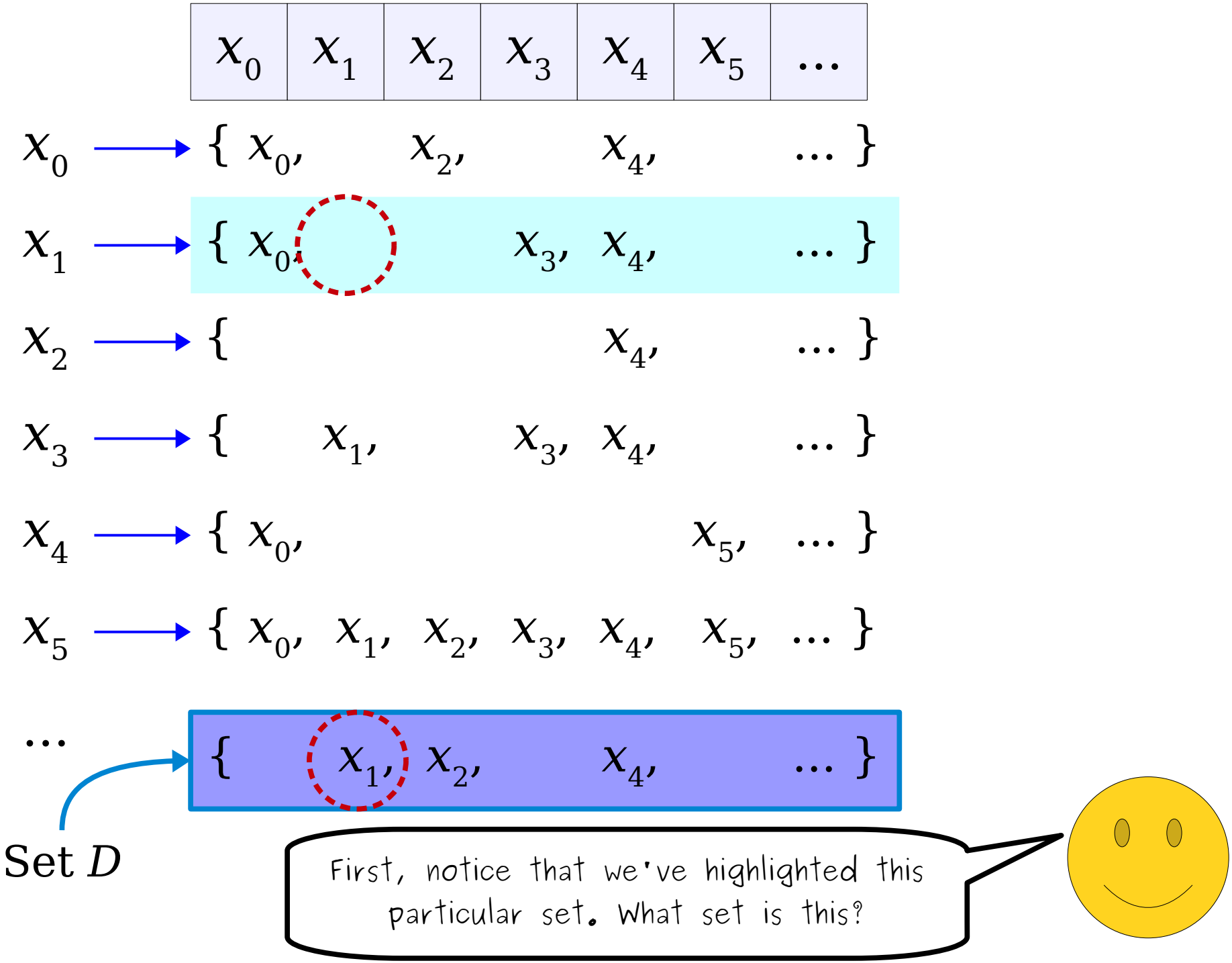
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



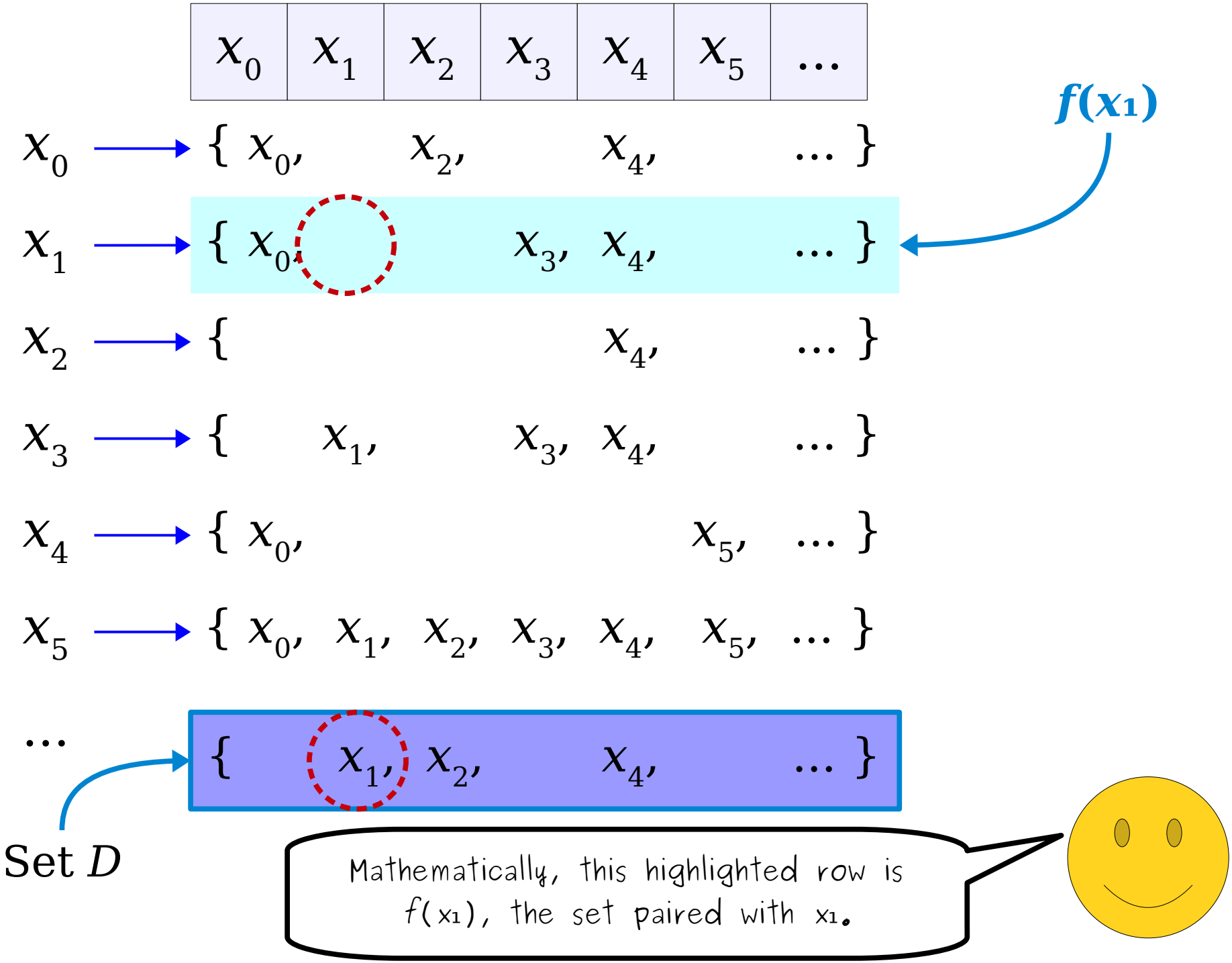
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



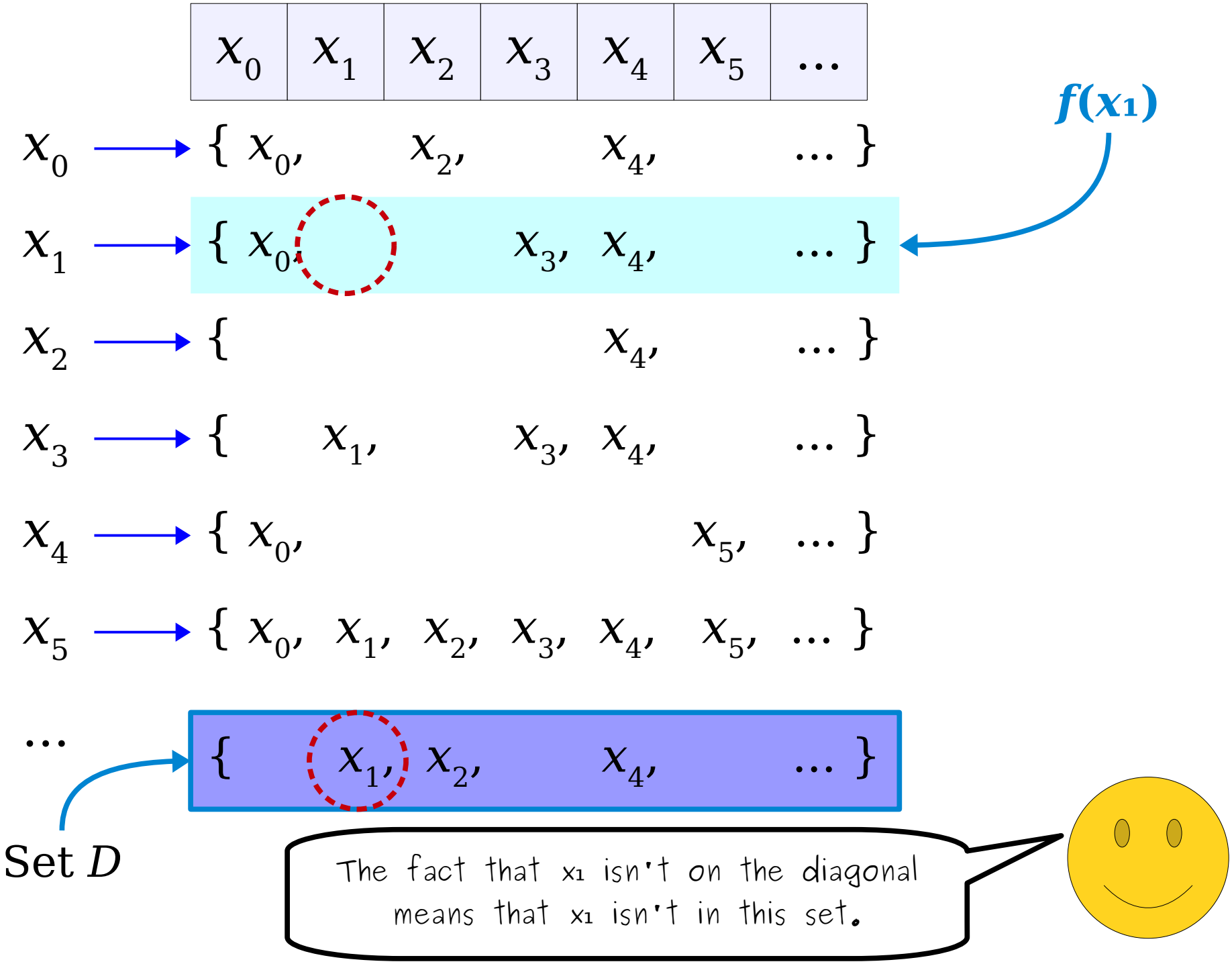
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



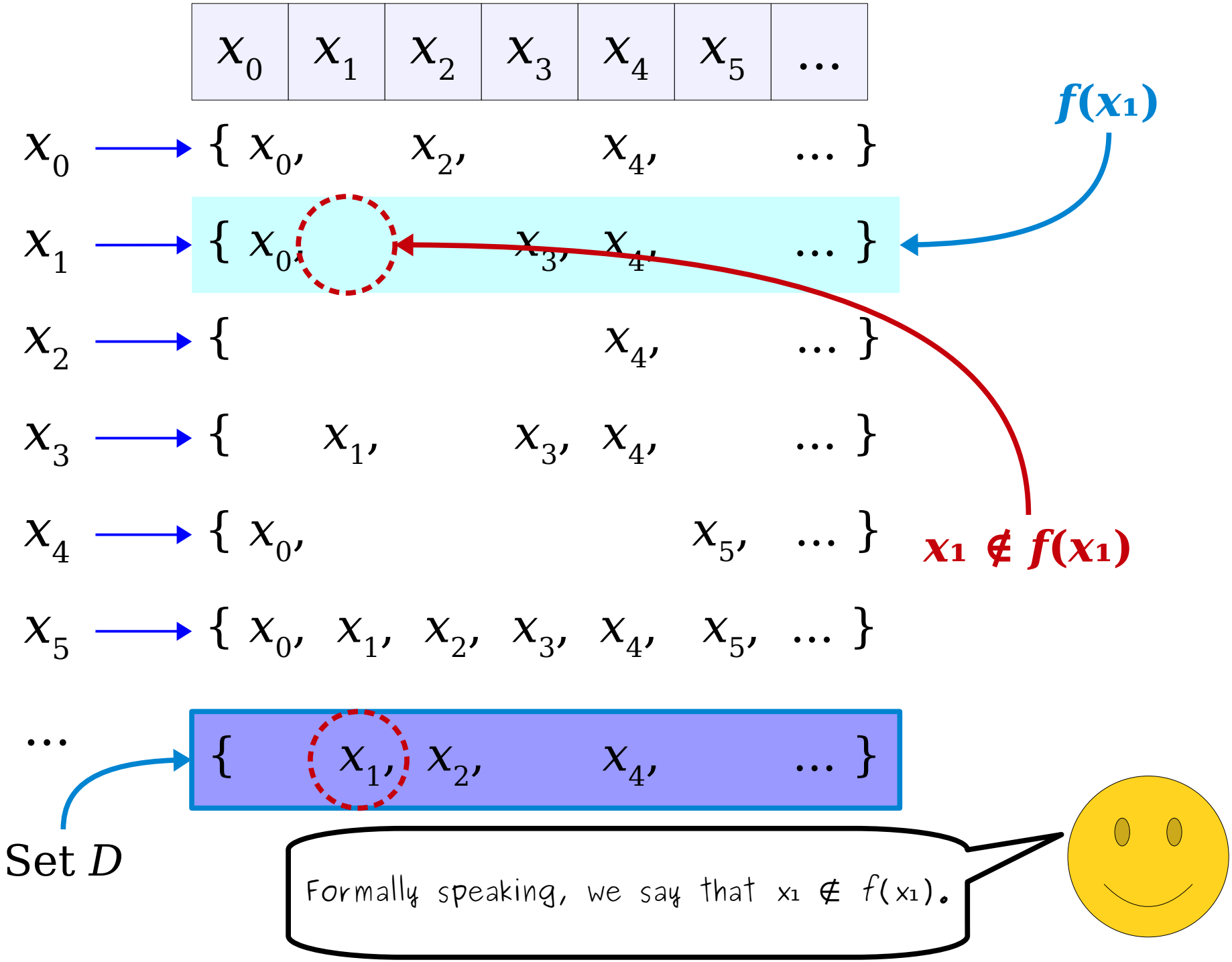
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



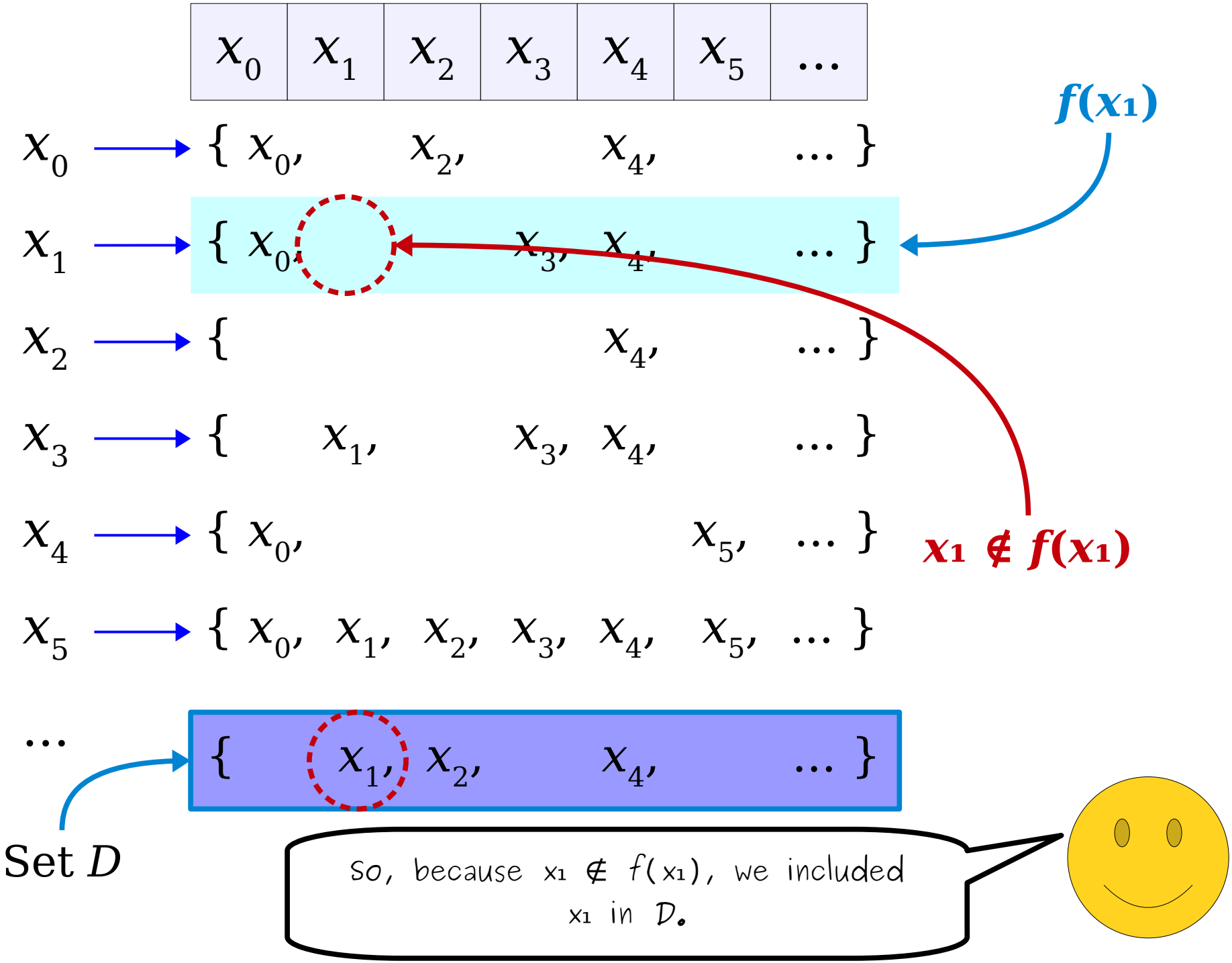
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

$x_3 \longrightarrow \{\quad x_1,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_4 \longrightarrow \{\ x_0,\quad x_5,\quad \ldots\ \}$

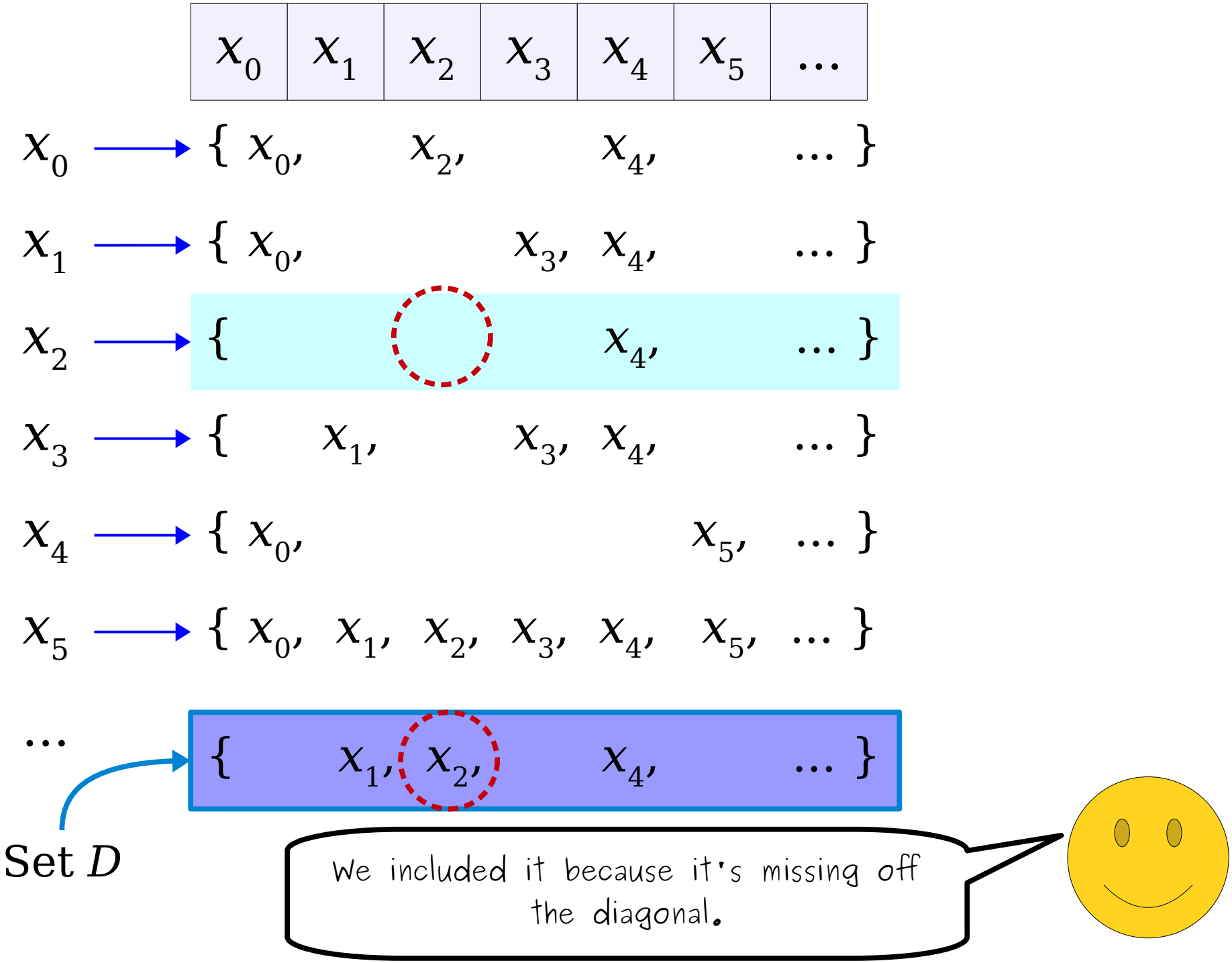
$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$\ldots$ $\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

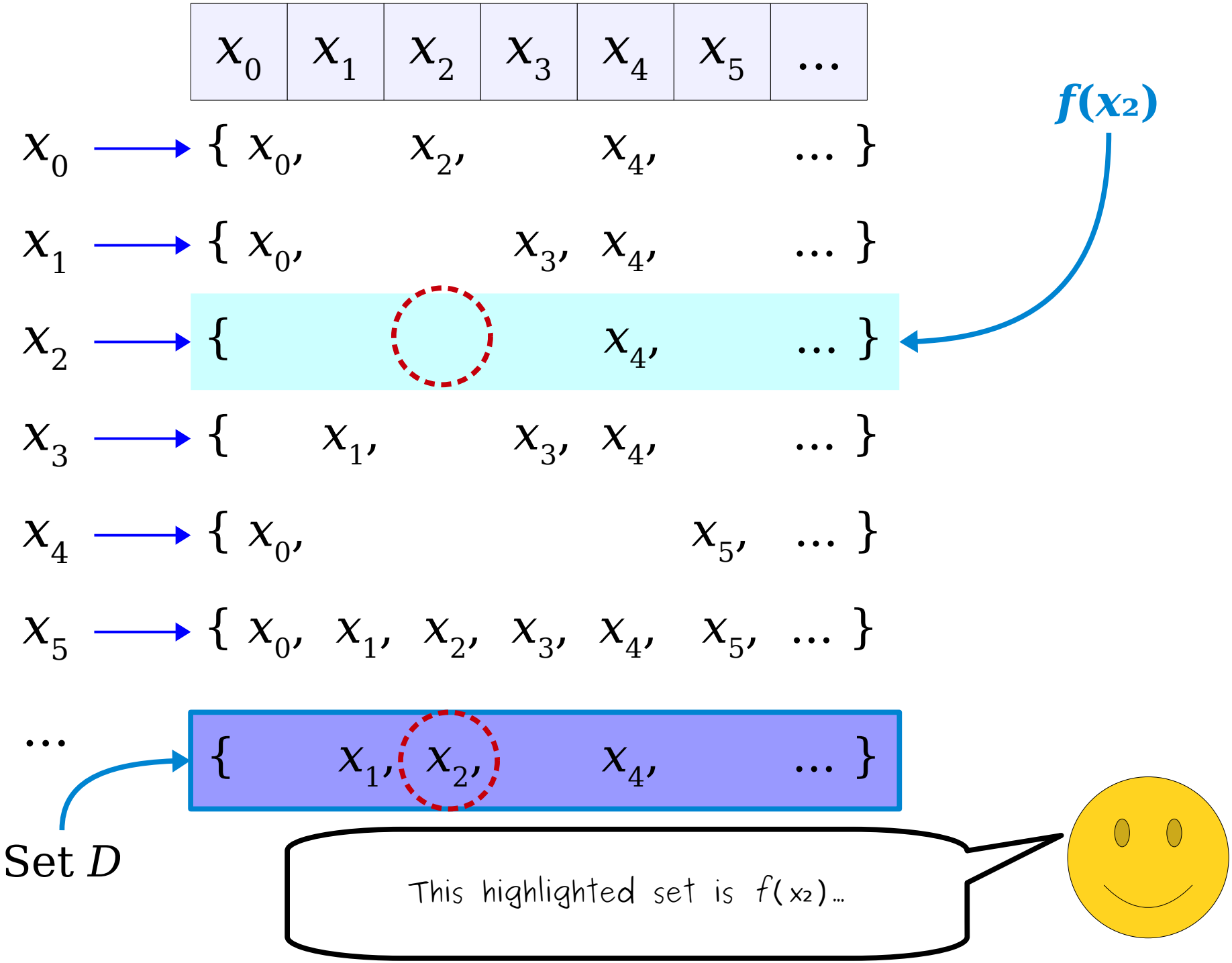
Set D

Similarly, let's take a look at why x_2 is in our set.

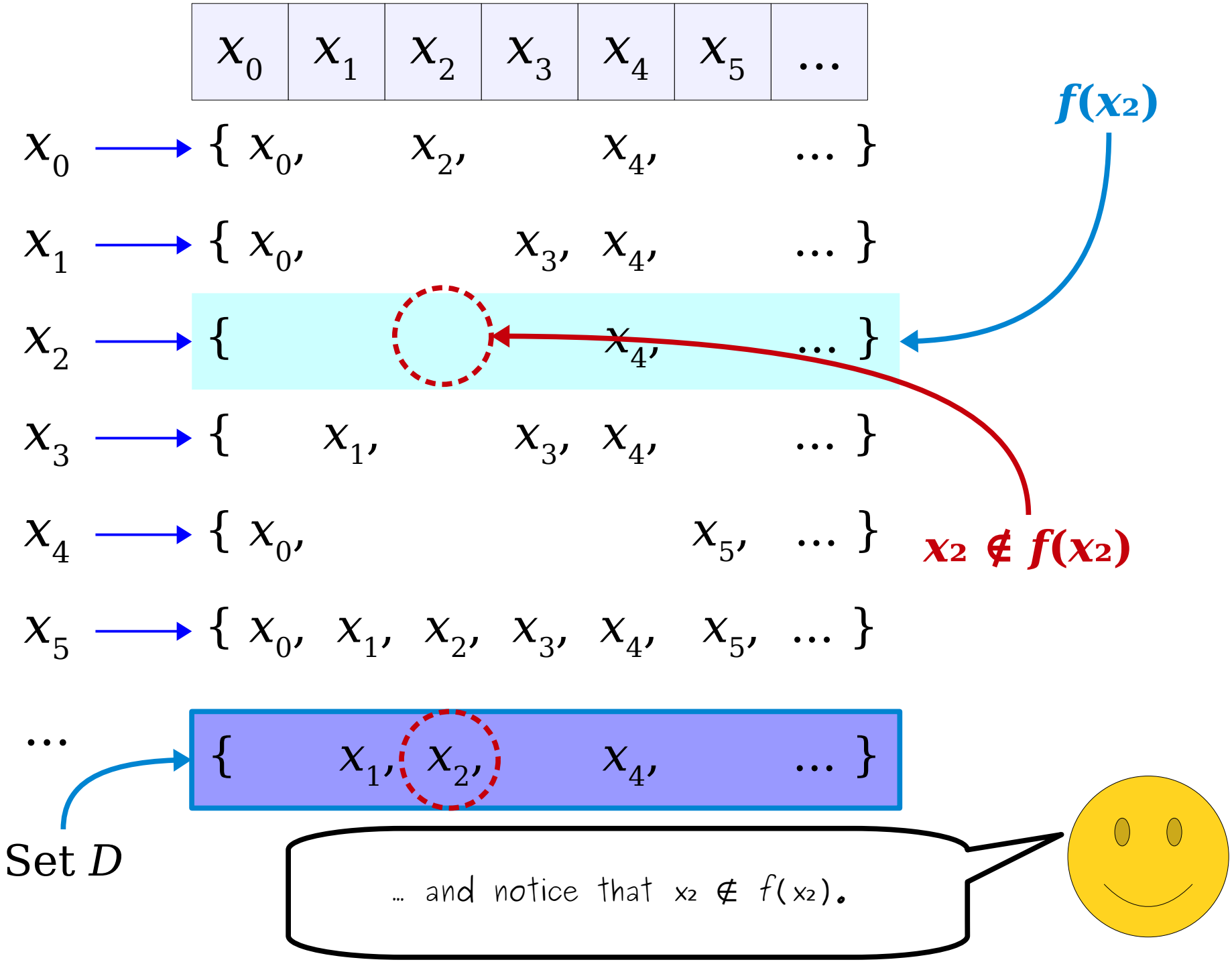
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



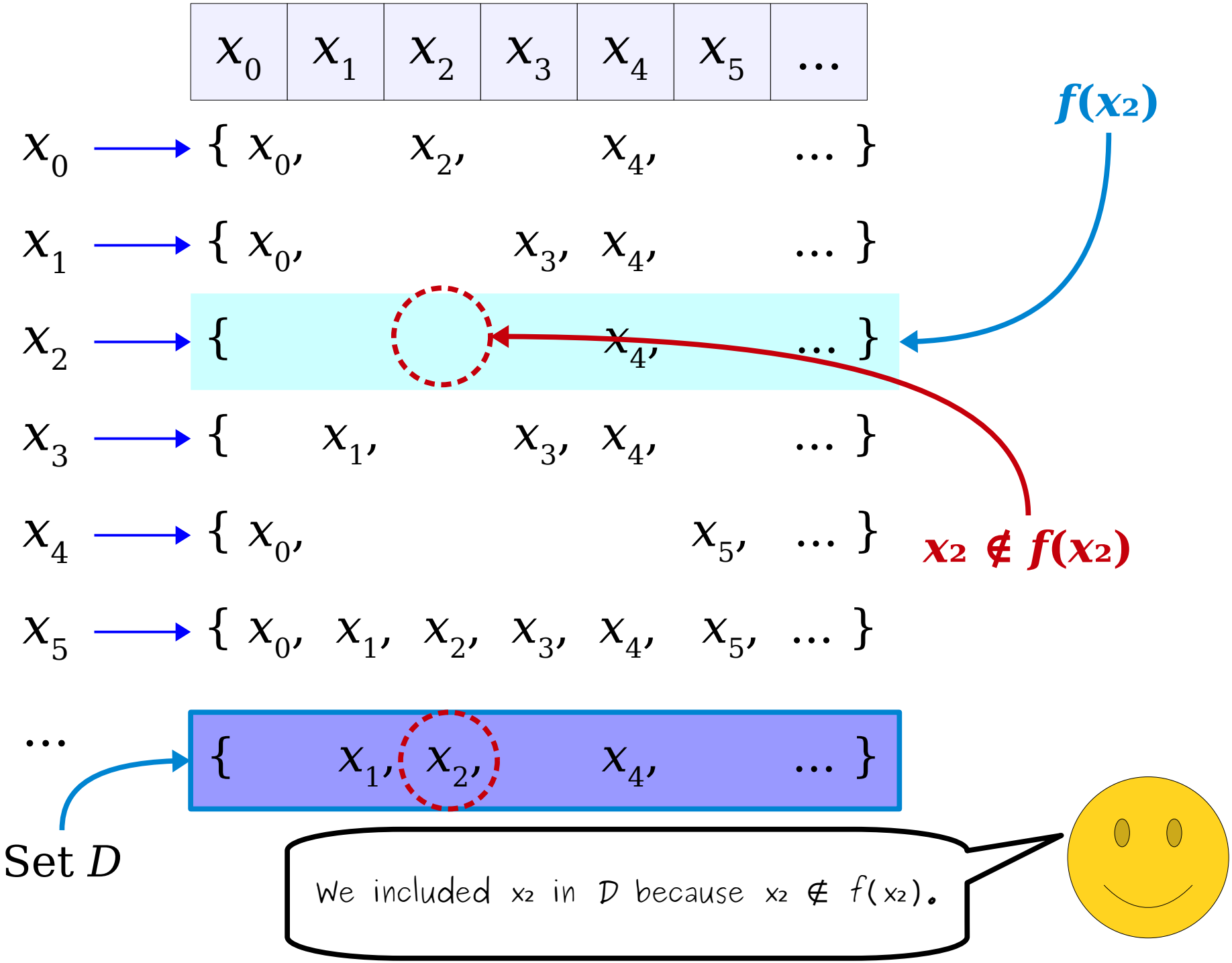
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



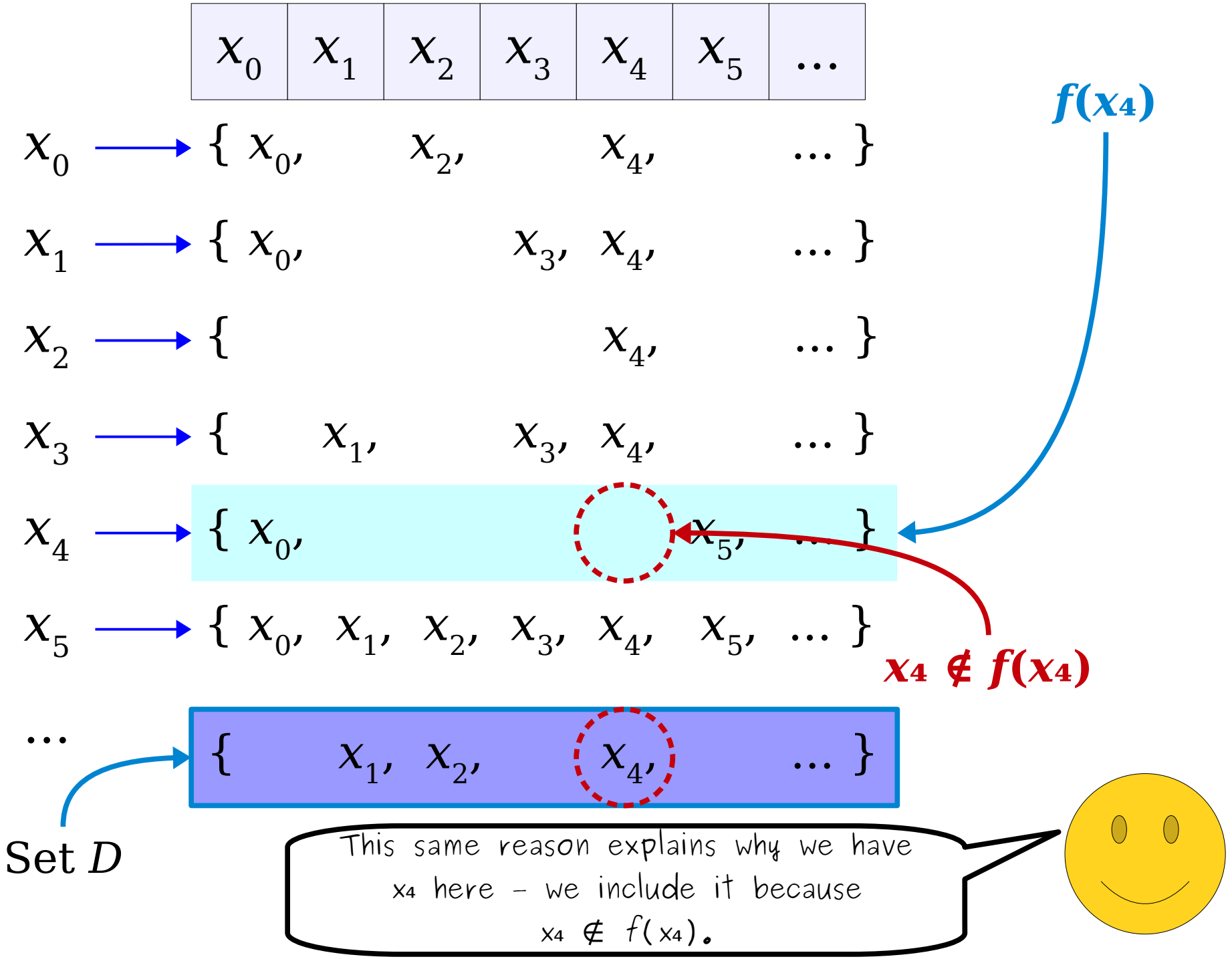
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

So it seems like we have a general pattern going here.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$

Set D $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Specifically, if we find an x where $x \notin f(x)$, then it ends up in D.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2,$

(That's actually really important, so make a note of this for later on.)

…

$\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

Specifically, if we find an x where $x \notin f(x)$, then it ends up in D.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

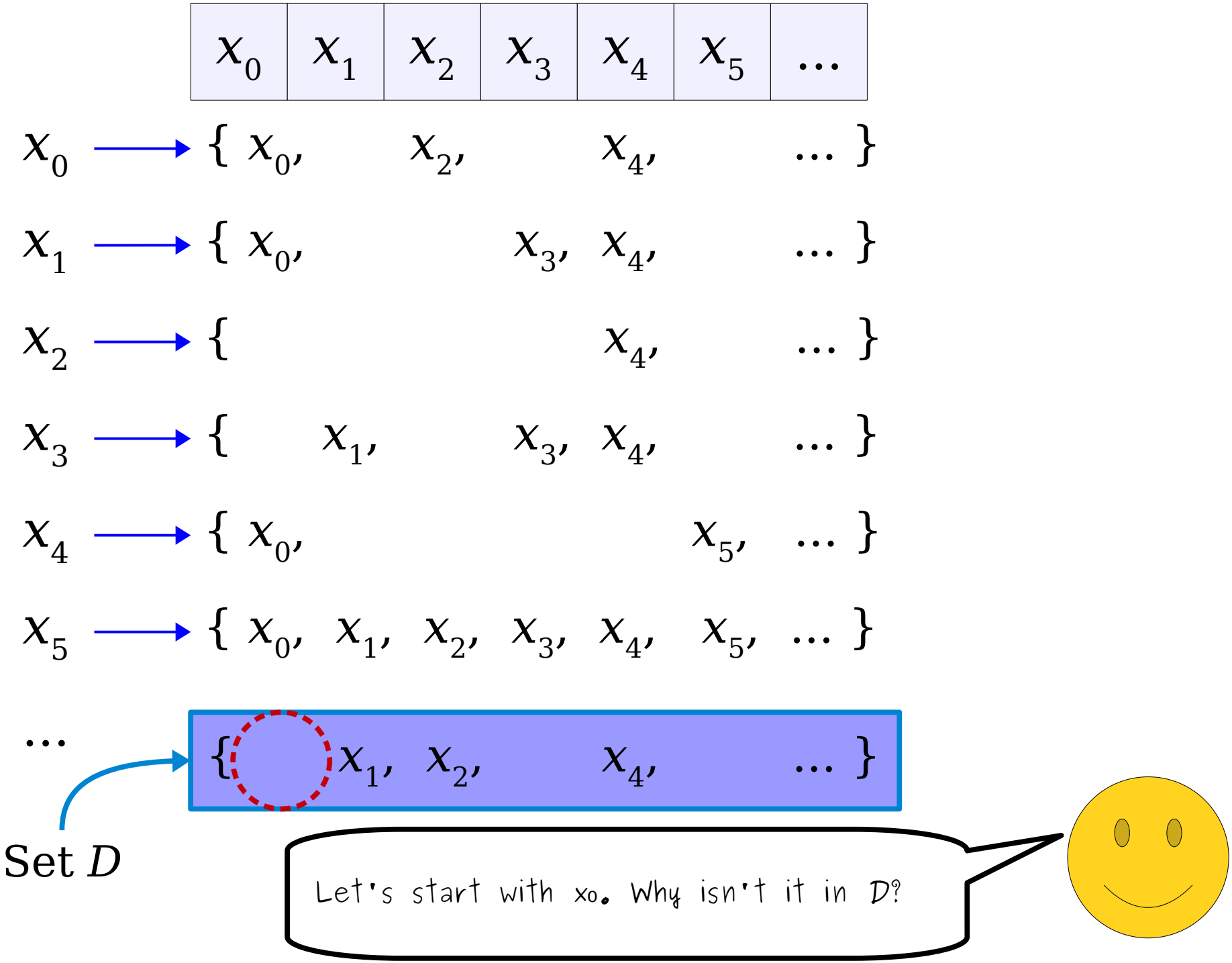
$\dots$

$\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

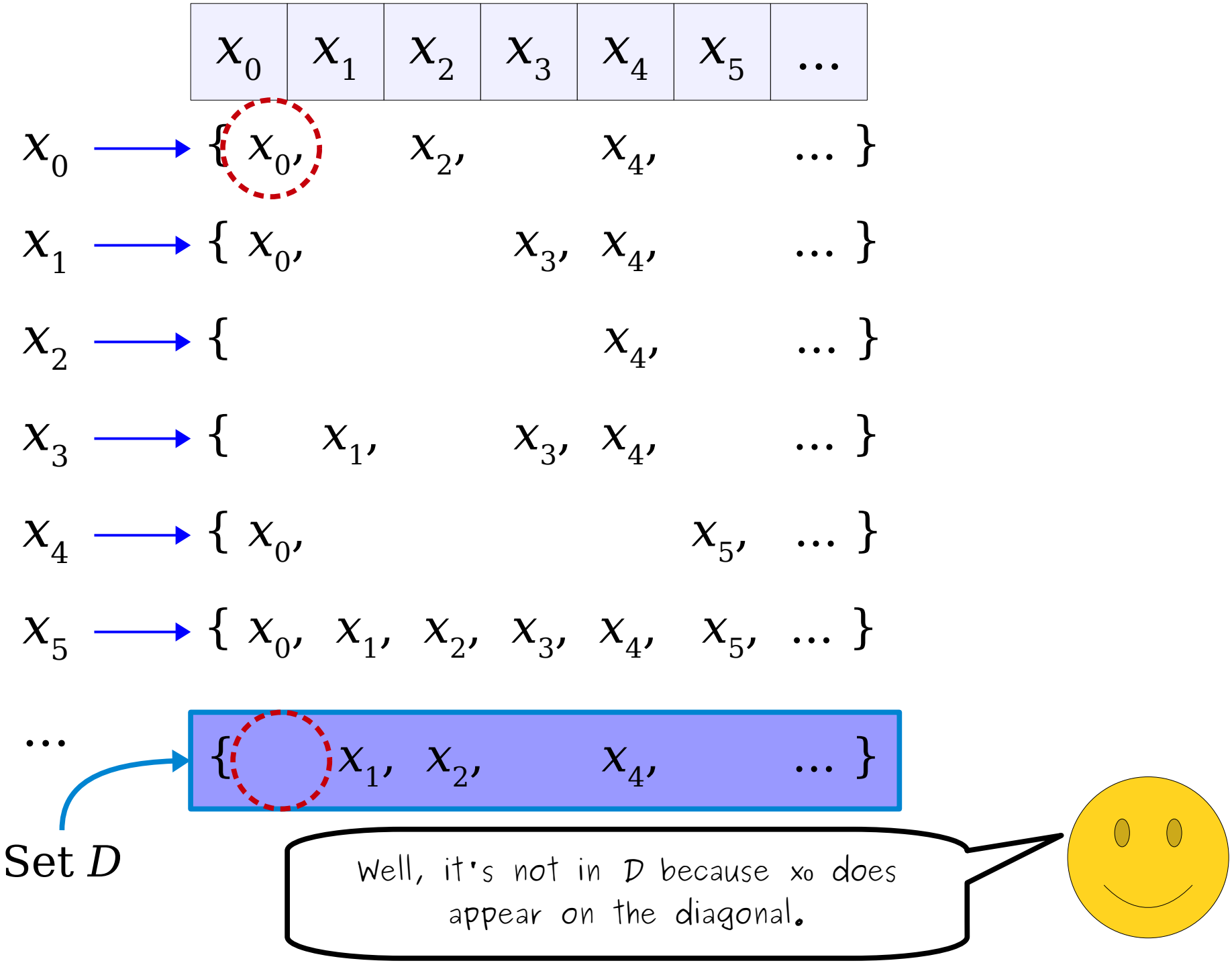
Set D

So what about the elements that aren't in our set D?

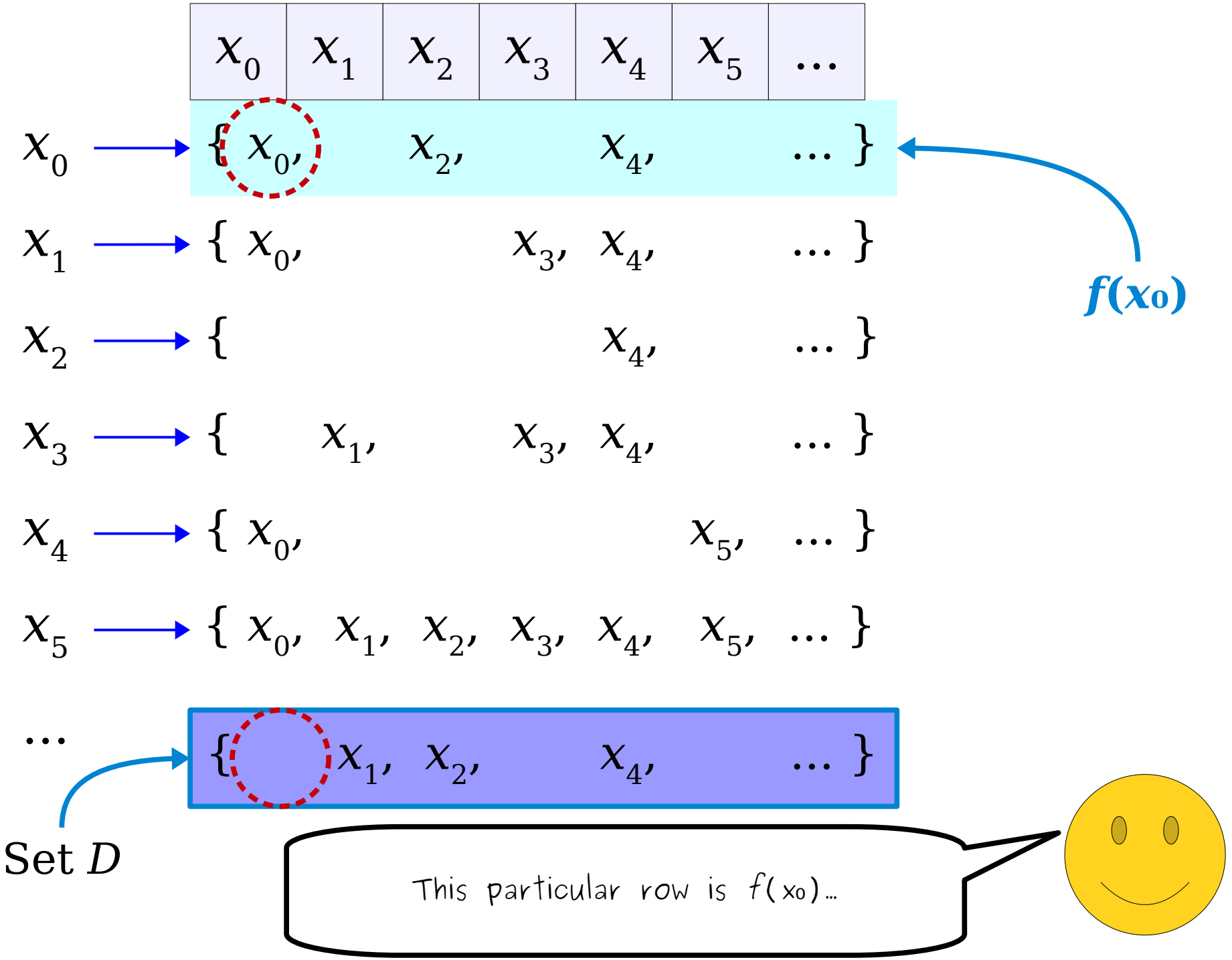
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



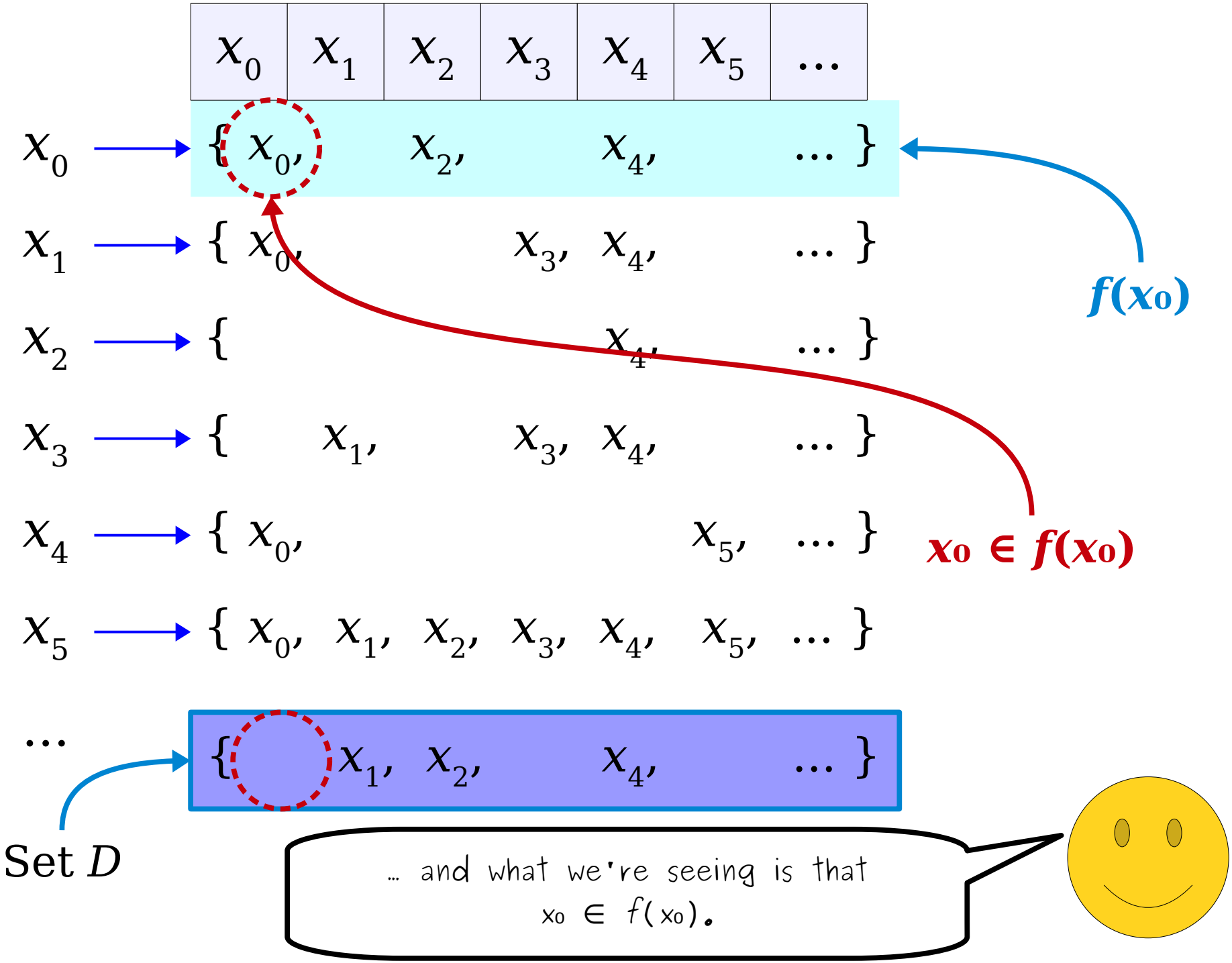
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



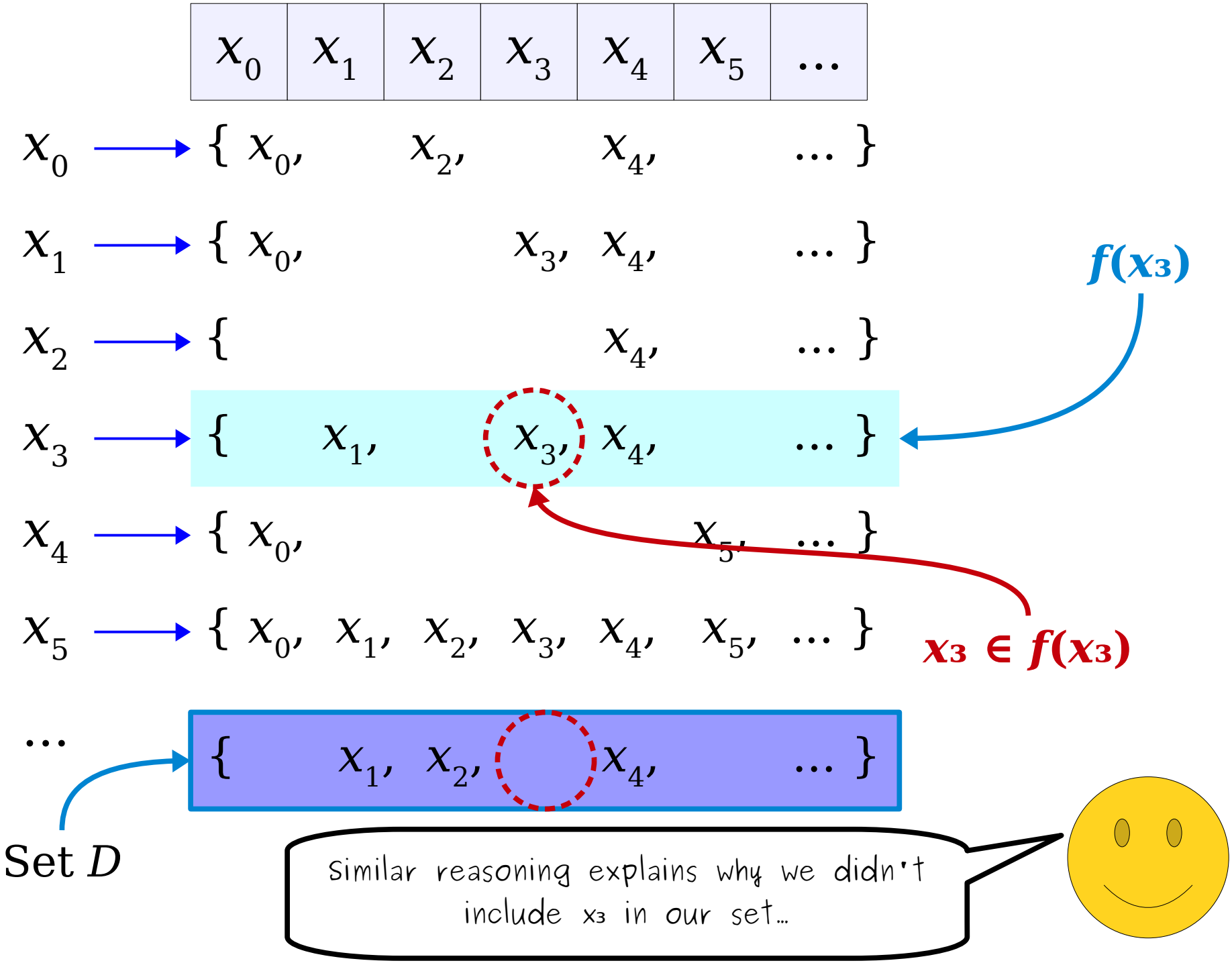
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



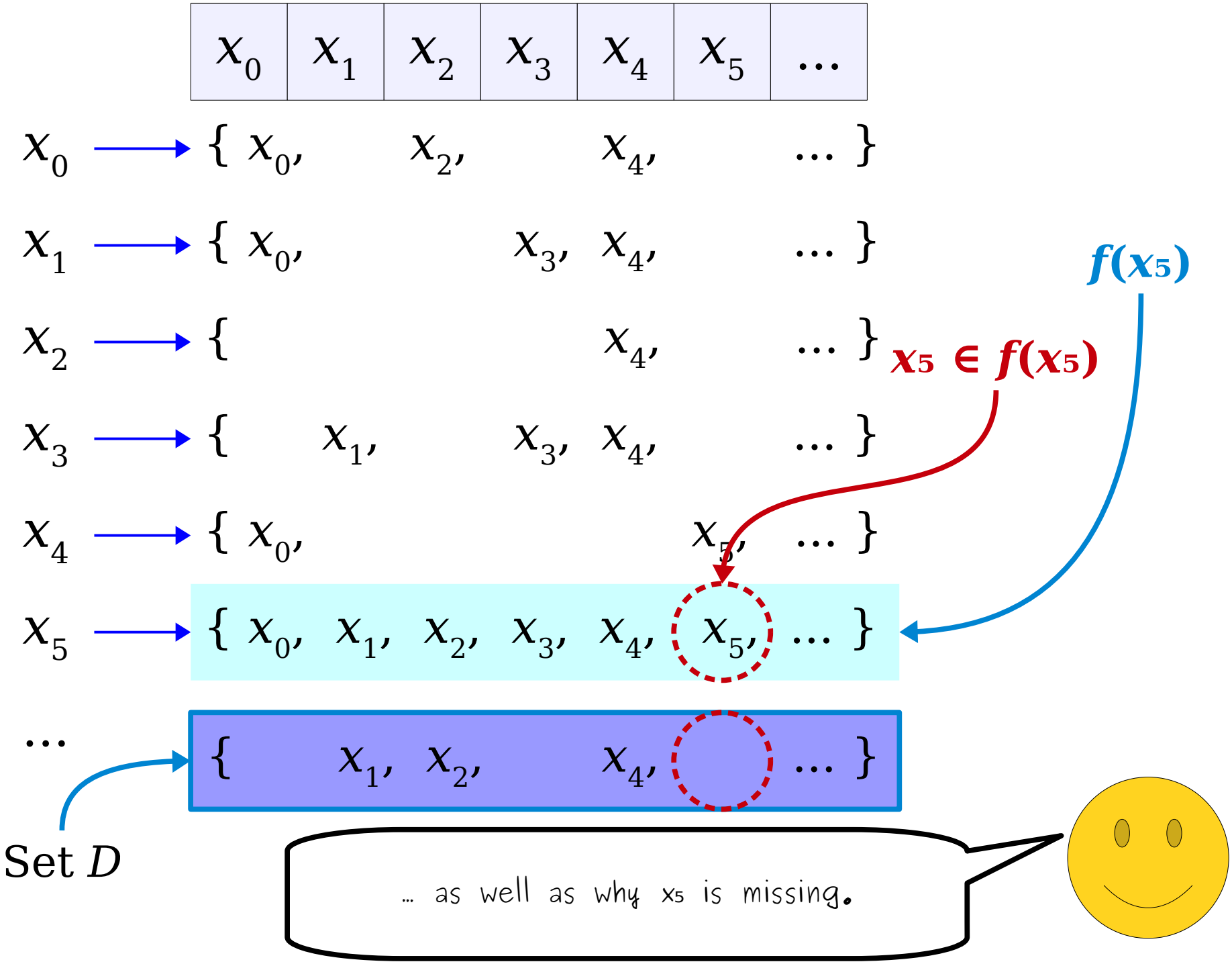
Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

So it seems like we have another general rule…

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad \dots \}$

$\dots$ $\{ \quad x_1, \quad x_2, \quad x_4, \quad \dots \}$

Set D

…the set D excludes every x where $x \in f(x)$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

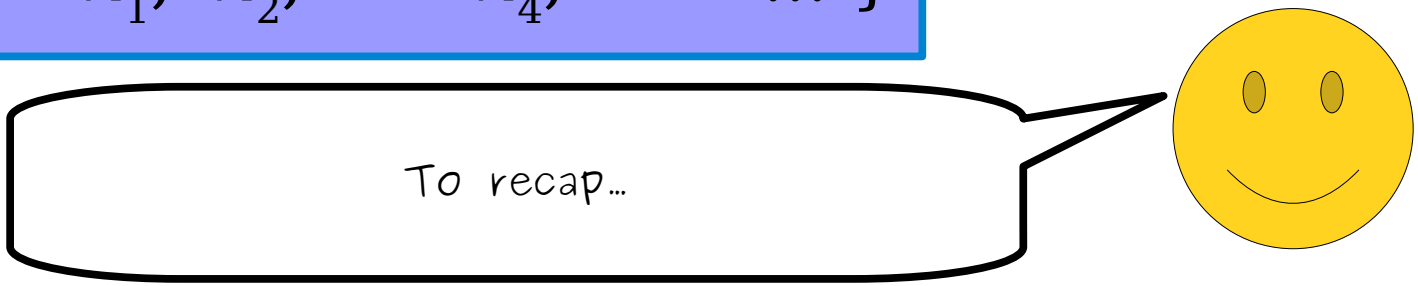
$x_3 \longrightarrow \{\quad x_1,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_4 \longrightarrow \{\ x_0,\quad x_5,\quad \ldots\ \}$

$x_5 \longrightarrow \{\ x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots\ \}$

$\ldots$ $\{\quad x_1,\ x_2,\quad x_4,\quad \ldots\ \}$

Set D



Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \dots \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \dots \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

If $x \notin f(x)$, then $x \in D$.

$\{ \quad x_1, x_2, \quad x_4, \quad \dots \}$

Set D

If we choose any x and see that x ∉ f(x), then x ∈ D.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

Set D → $\{ \quad x_1, x_2, \quad x_4, \quad \dots \}$

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

Similarly, if we choose any x and find that $x \in f(x)$, then $x \notin D$.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

$x_2 \longrightarrow \{ \quad x_4, \quad \dots \}$

$x_3 \longrightarrow \{ \quad x_1, \quad x_3, \quad x_4, \quad \}$

$x_4 \longrightarrow \{ x_0, \quad x_5, \quad \}$

$x_5 \longrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

If $x \notin f(x)$, then $x \in D$.
If $x \in f(x)$, then $x \notin D$.

$\{ \quad x_1, x_2, \quad x_4, \quad \dots \}$

Set D

Collectively, this lets us define our set D purely mathematically.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{\ x_0,\quad x_2,\quad x_4,\quad \ldots\ \}$

$x_1 \longrightarrow \{\ x_0,\quad x_3,\quad x_4,\quad \ldots\ \}$

$x_2 \longrightarrow \{\quad x_4,\quad \ldots\ \}$

$x_3 \longrightarrow \{\quad x_1,\quad x_3,\quad x_4,\quad$

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If $x \notin f(x)$, then $x \in D$.
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$D = \{\ x \in S \mid x \notin f(x)\ \}$

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Set D

Specifically:

$D = \{\ x \in S \mid x \notin f(x)\ \}$

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x_0	x_1	x_2	x_3	x_4	x_5	…

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Take a minute to see why this is – it's a consequence of the two above rules.

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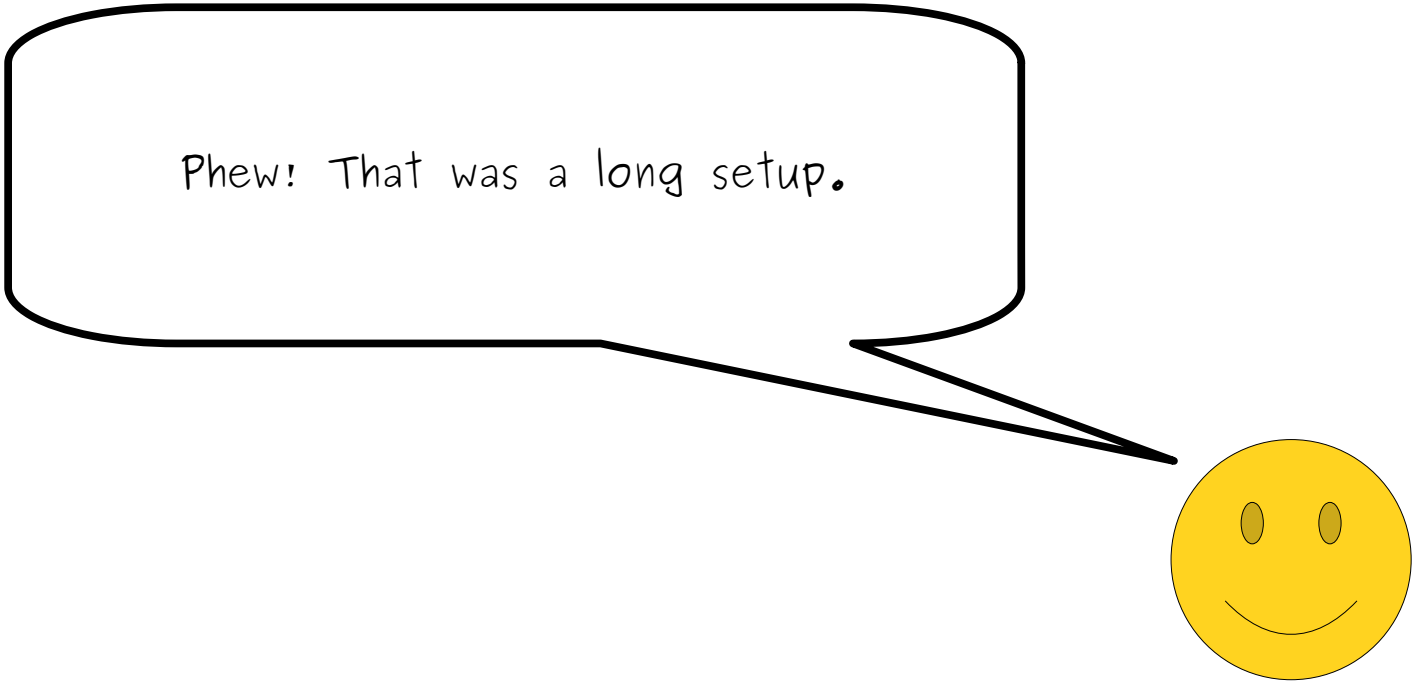

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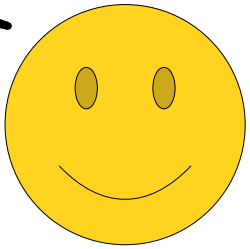
The whole point of that visual exercise was to figure out a mathematical way of describing that diagonal set.



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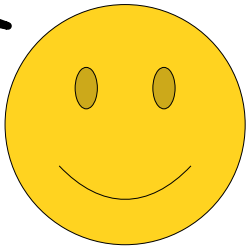
At this point, we actually don't need to draw any more pictures. Everything we need is summarized by the above symbolic notation.



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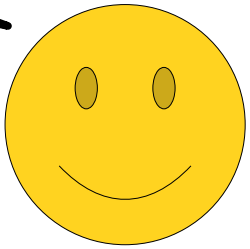
In fact, just knowing this particular choice of D allows us to write a formal proof that $|S| \neq |\wp(S)|$!



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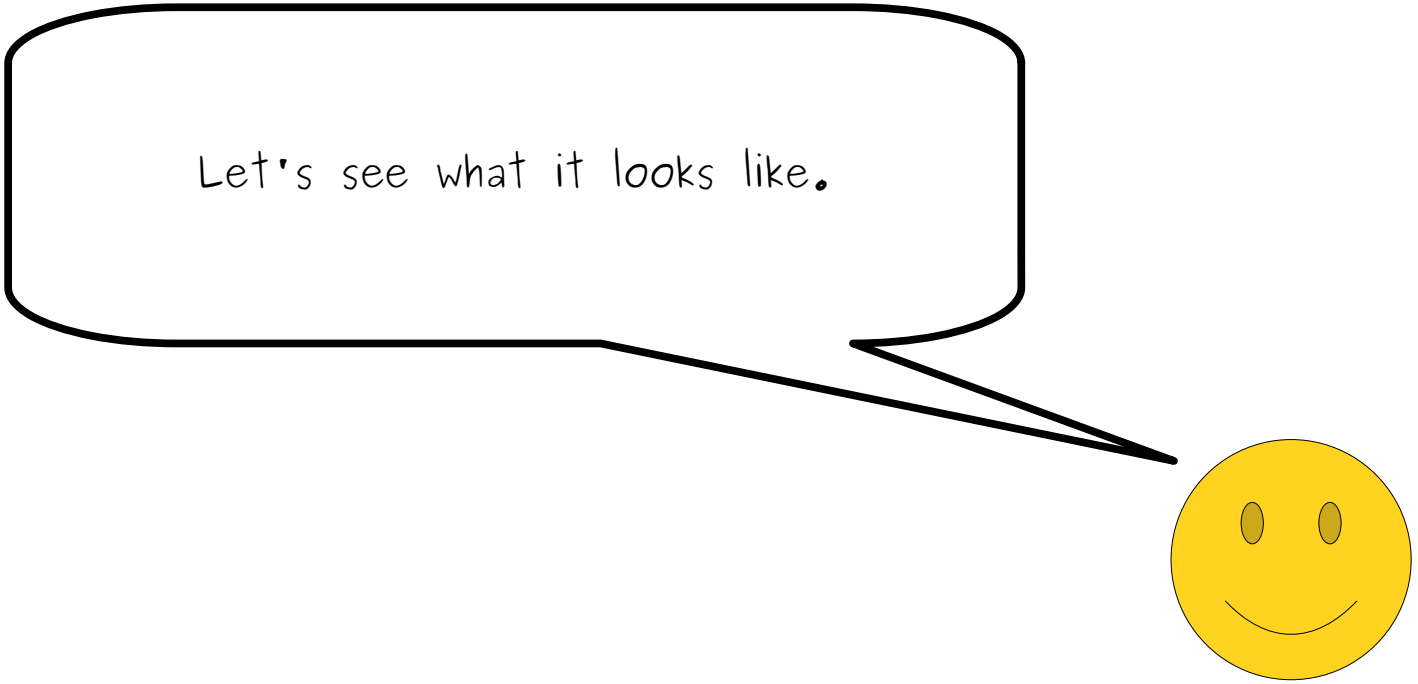

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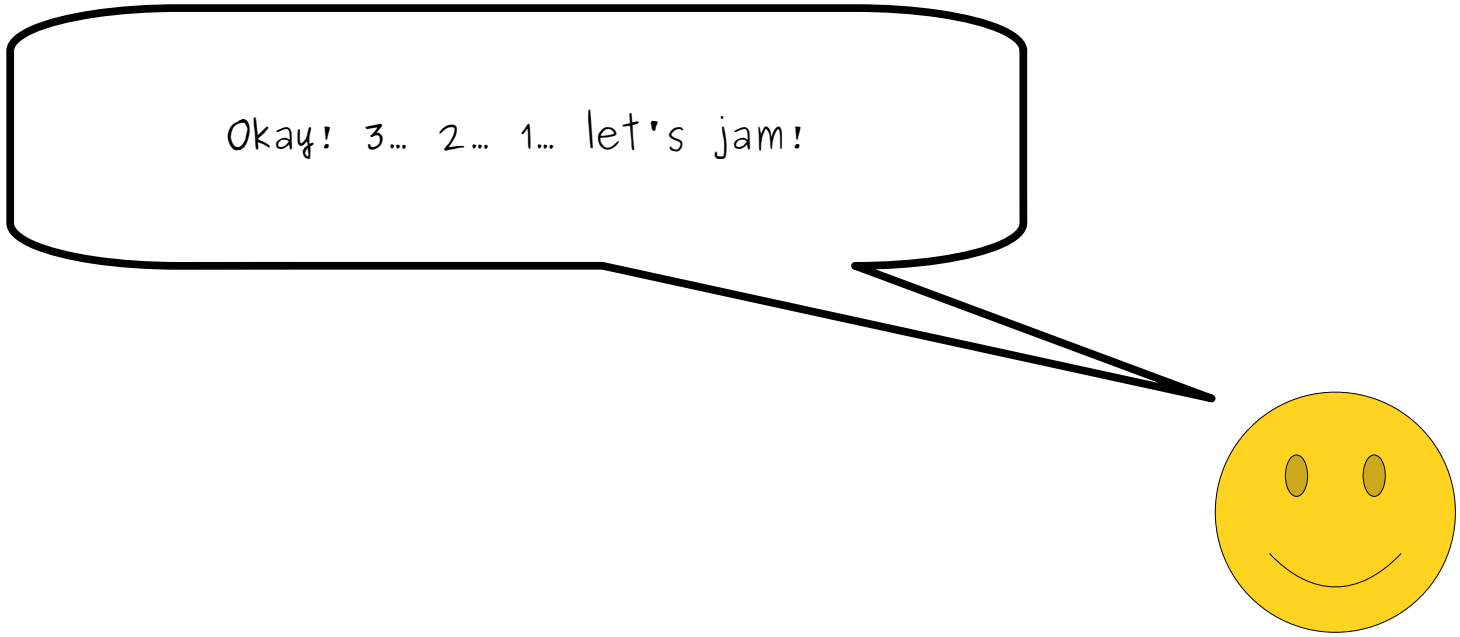

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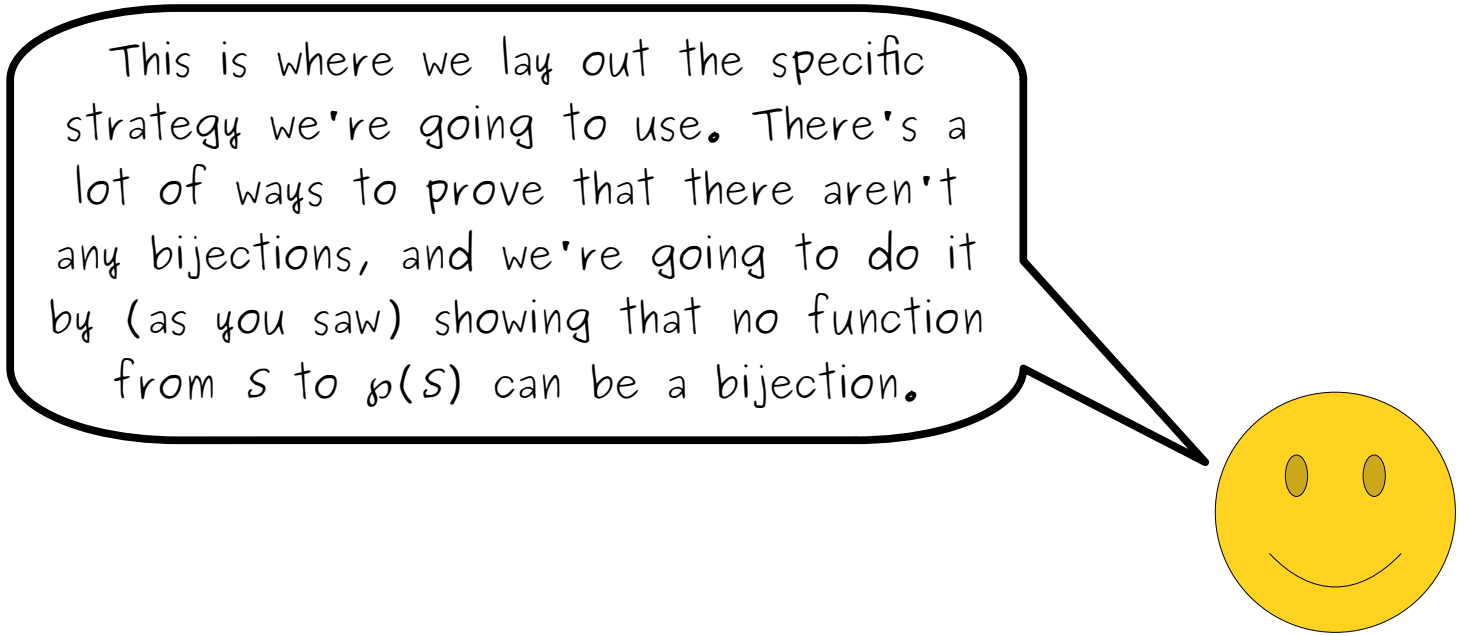

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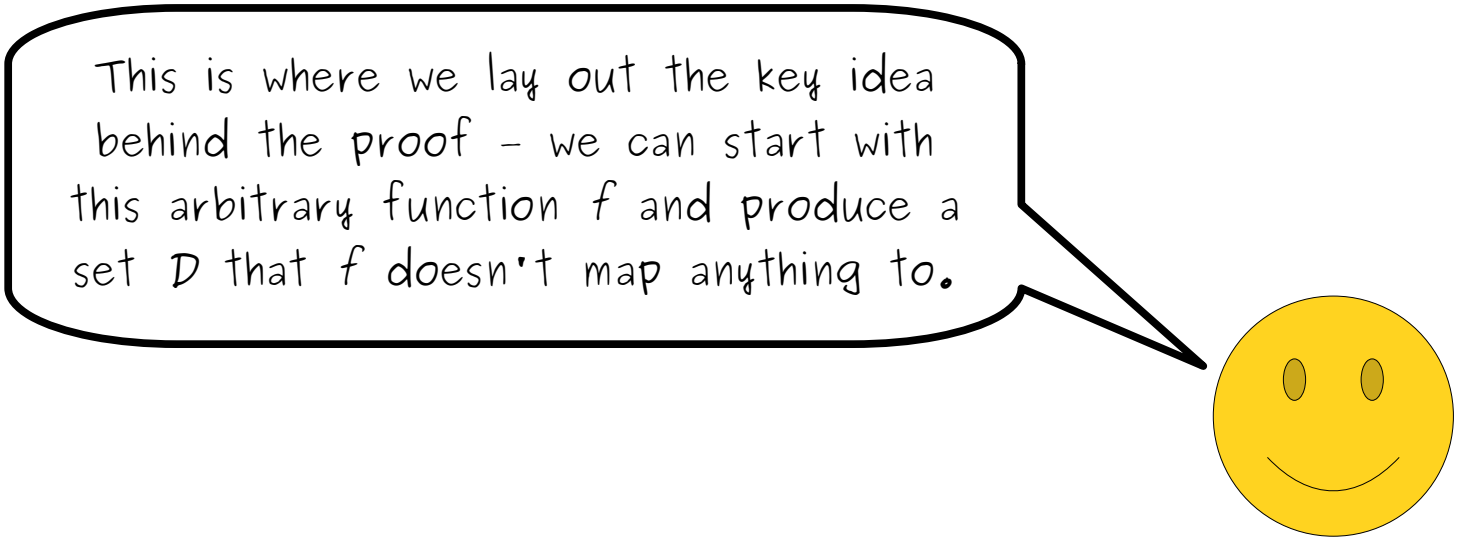

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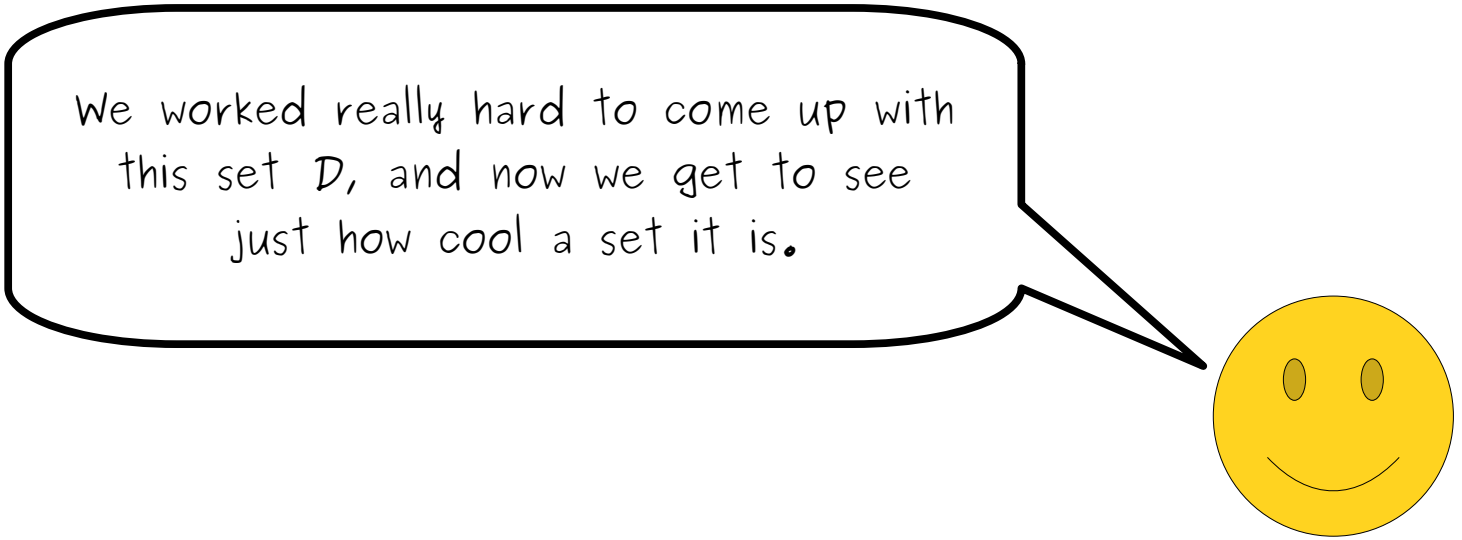

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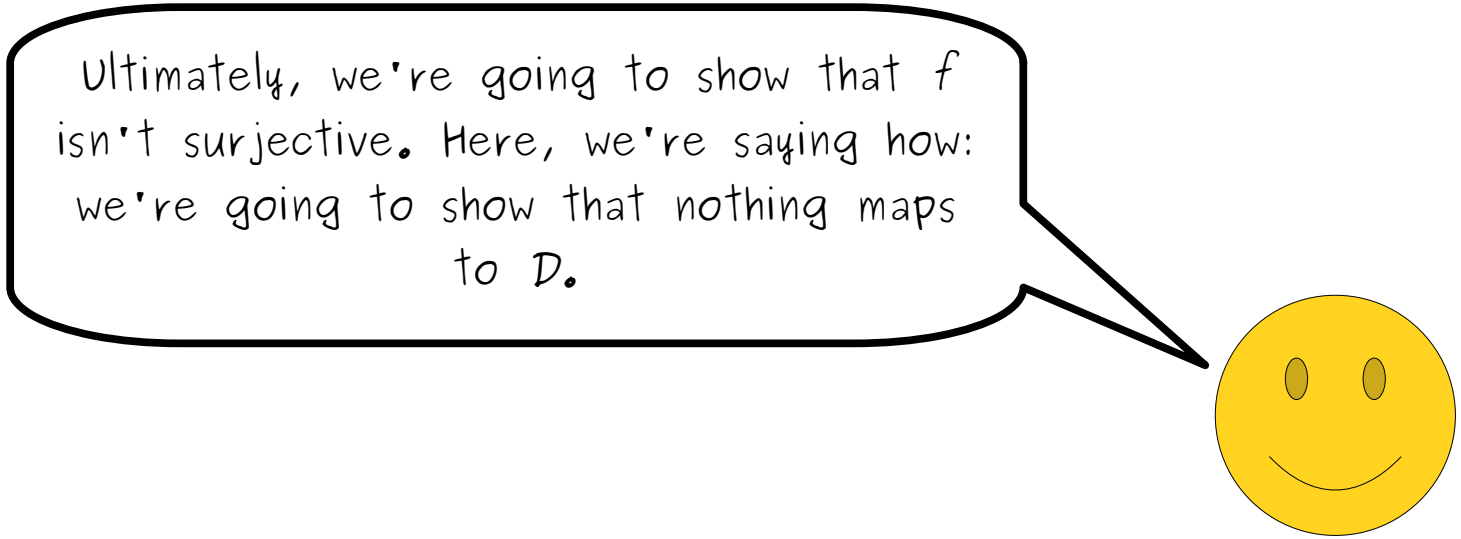

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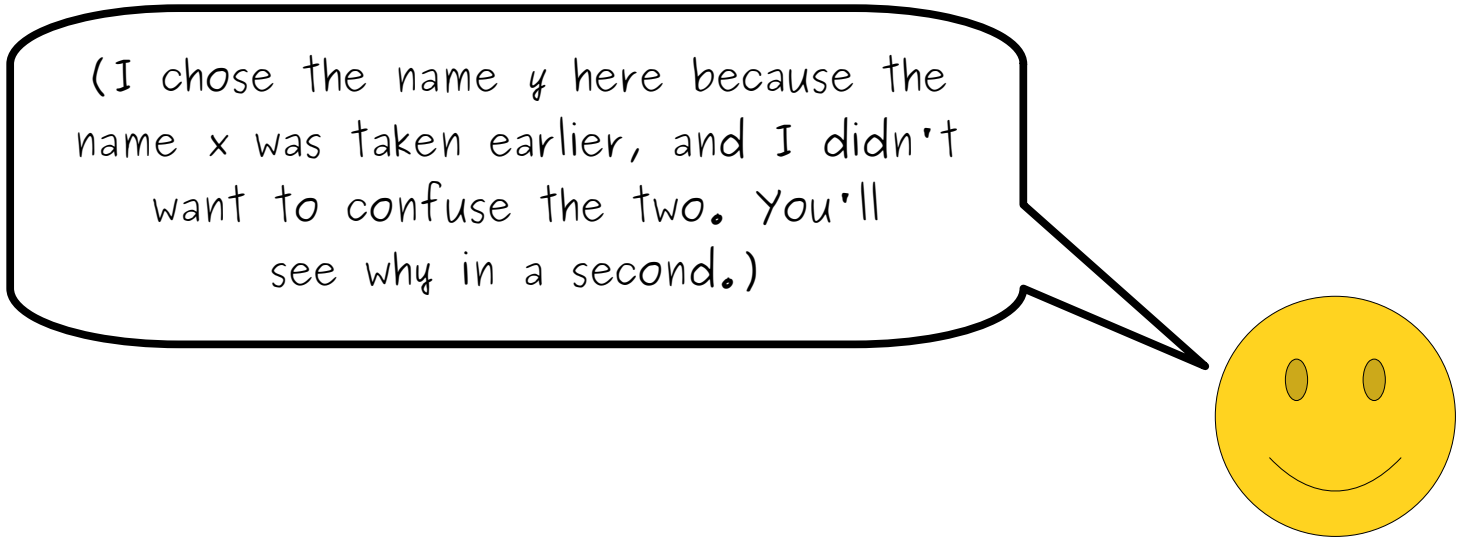

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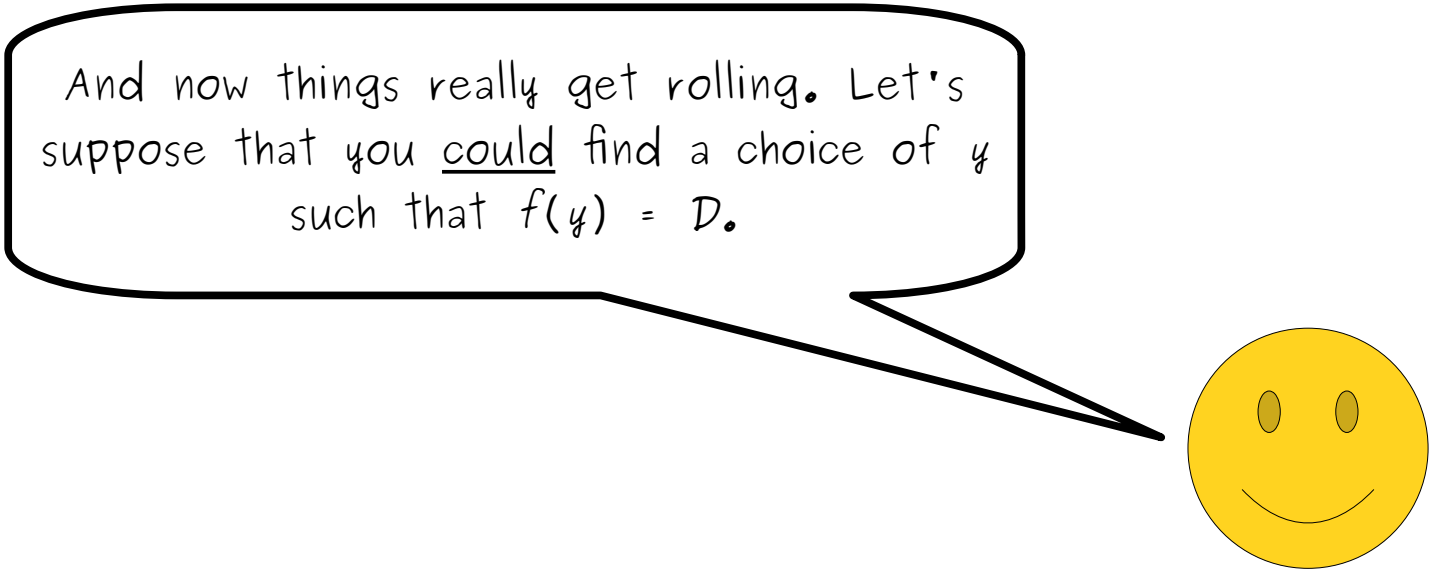

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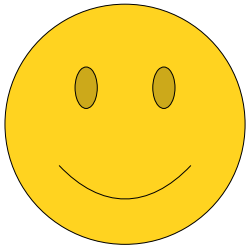
Our intuition behind building the set D was to get a contradiction between whether $y \in f(y)$ and whether $y \in D$. Specifically, if $y \notin f(y)$, then $y \in D$, and if $y \in f(y)$, then $y \notin D$. This is just the mathematical way of talking about what happens on the diagonal.



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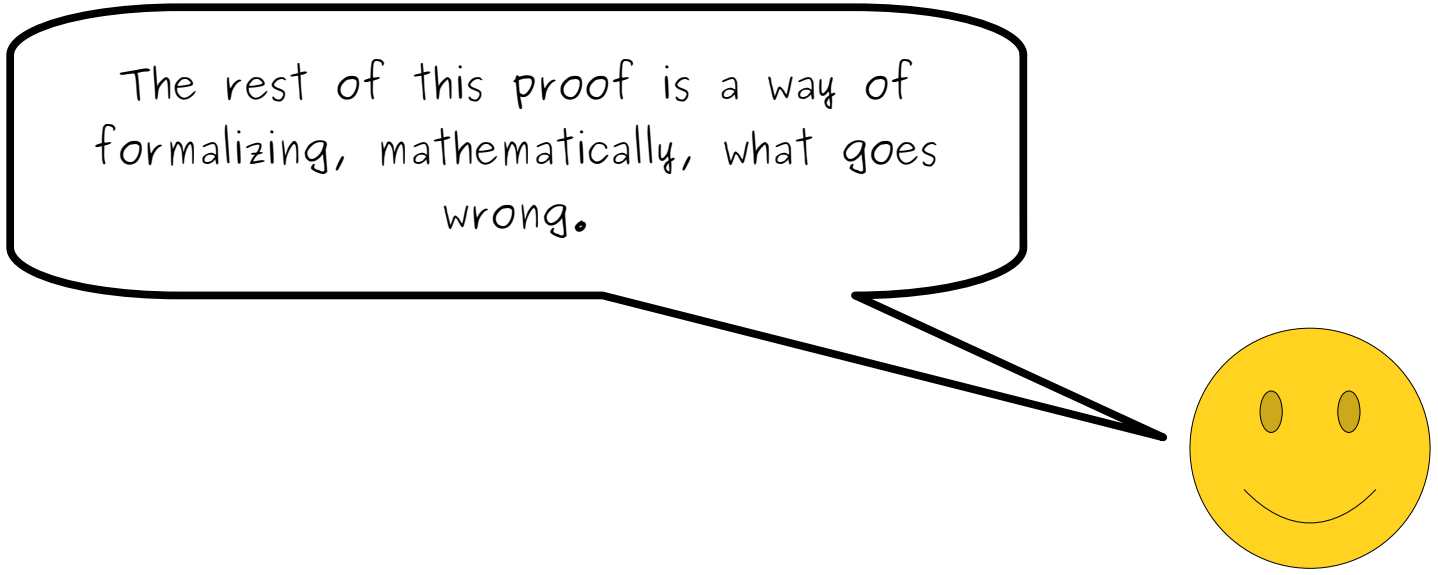

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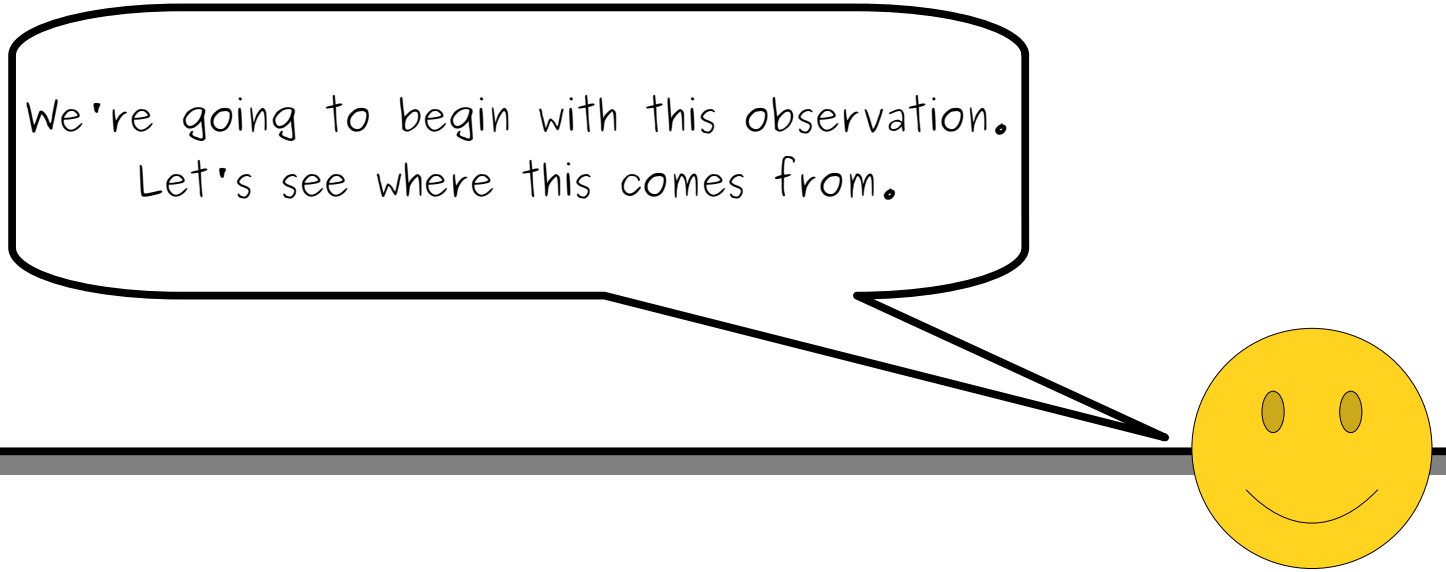

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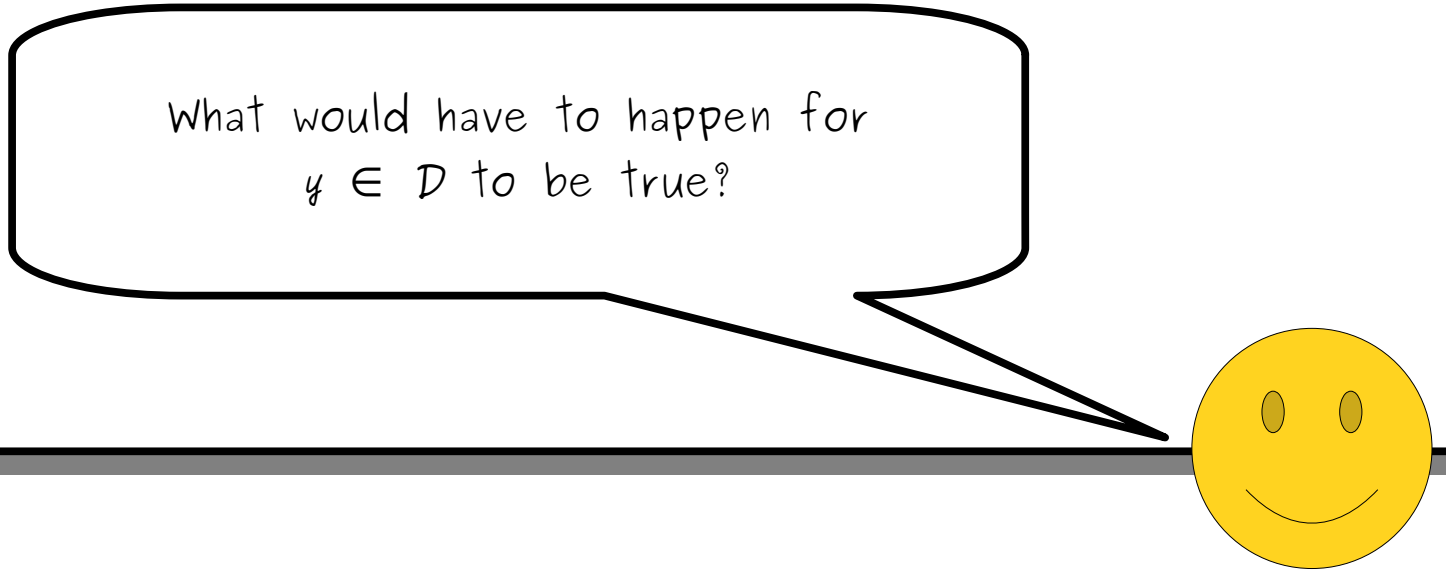

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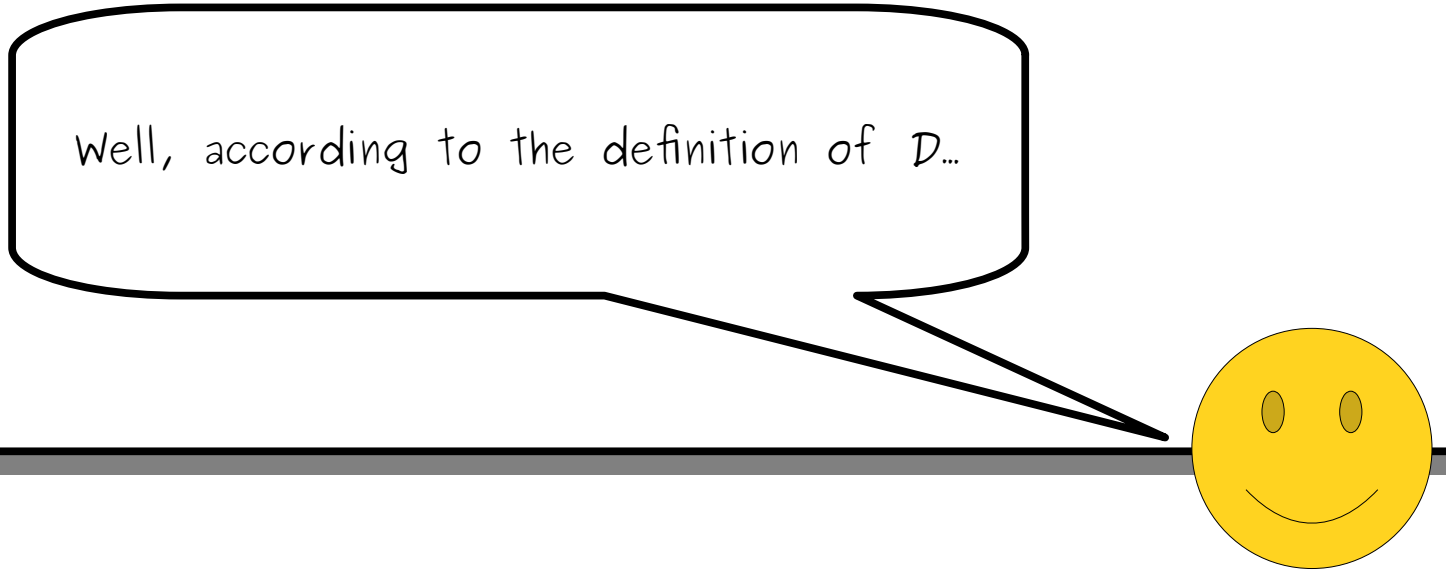

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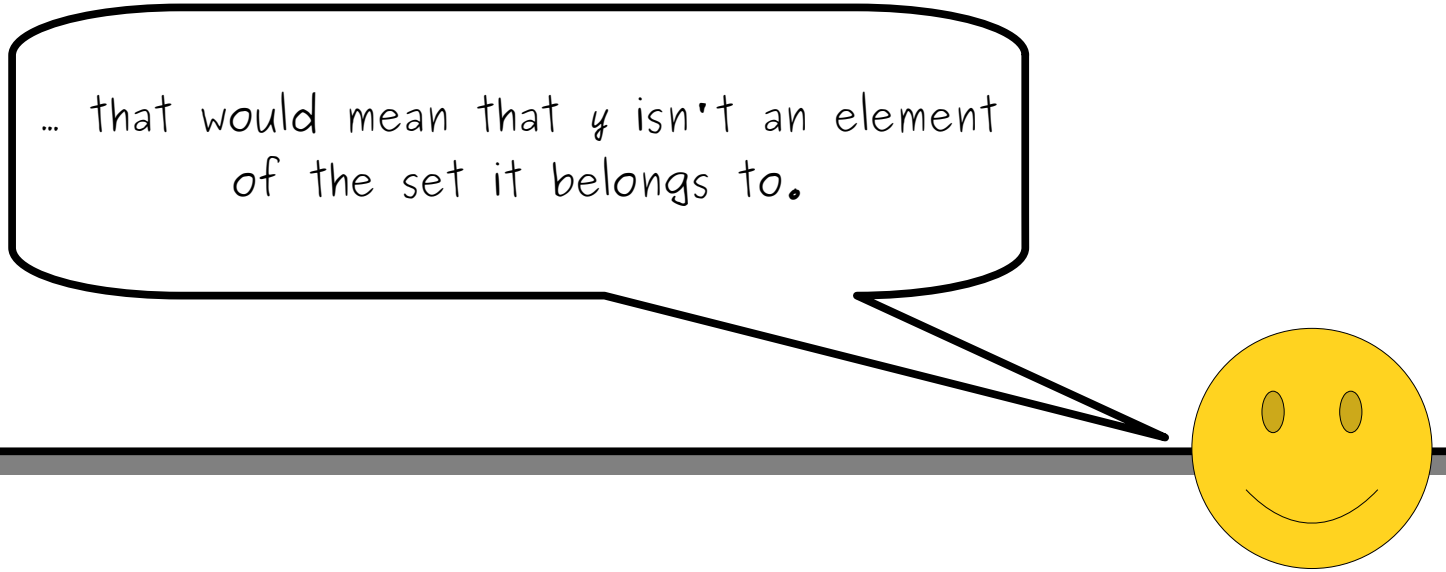

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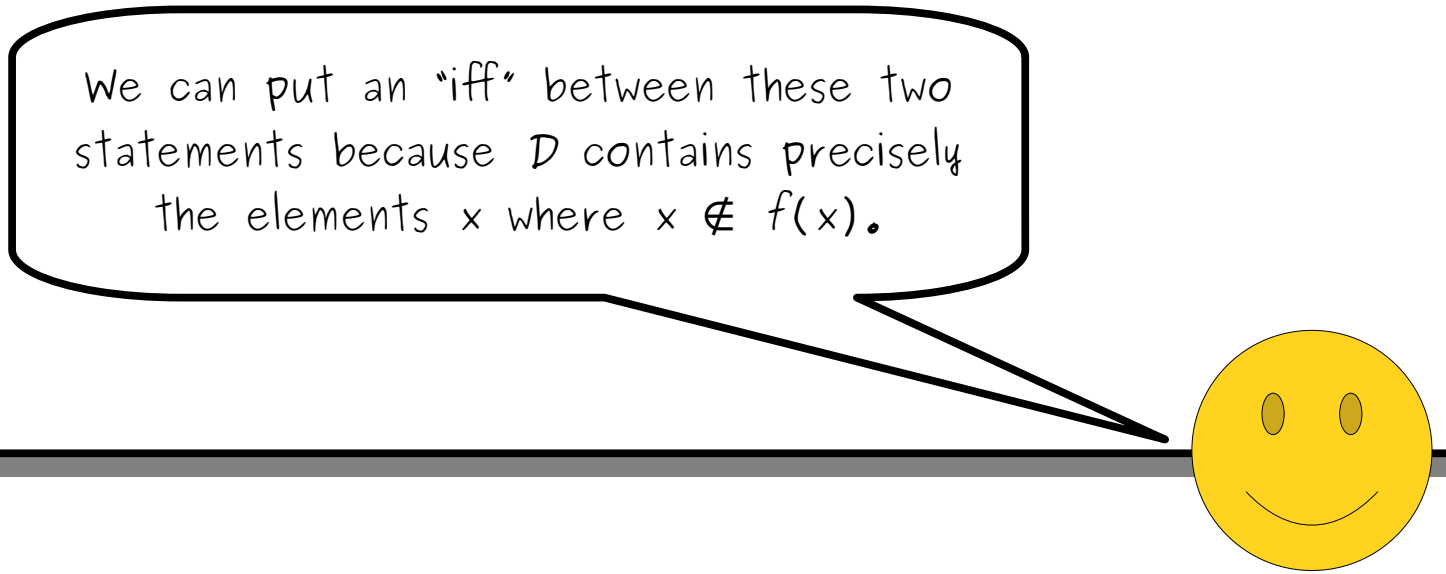

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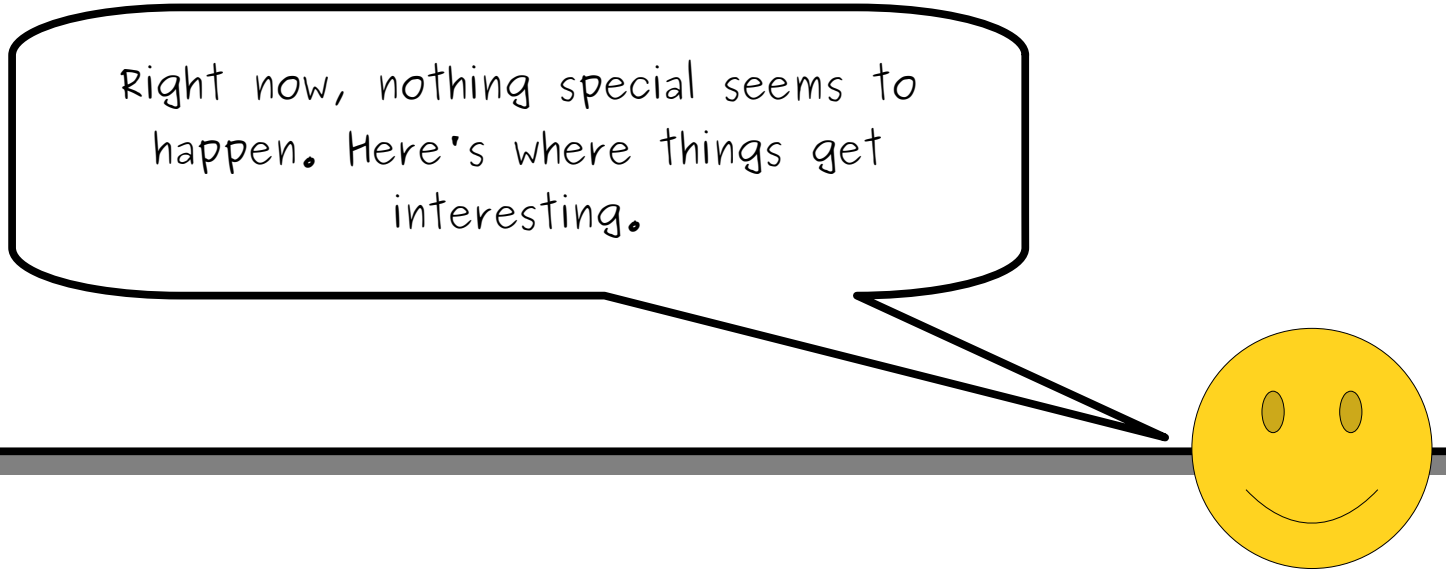

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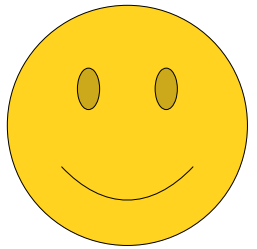
By assumption, $f(y) = D$. Combined with (2), this tells us

$$y \in D \text{ if and only if } y \notin D. \qquad (3)$$

We specifically assumed that $f(y) = D$. That means we can rewrite statement (2) as shown here. If you're wondering why we can do this…



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… notice that we're just substituting D for $f(y)$ in the right-hand side of this statement.

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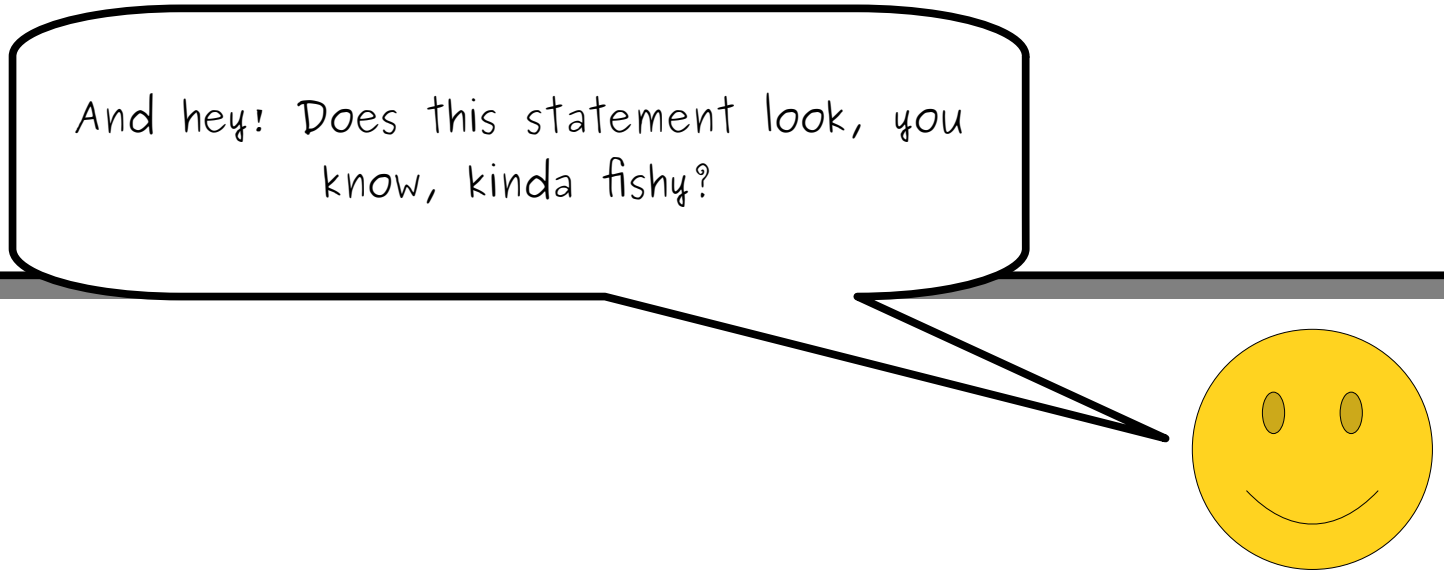

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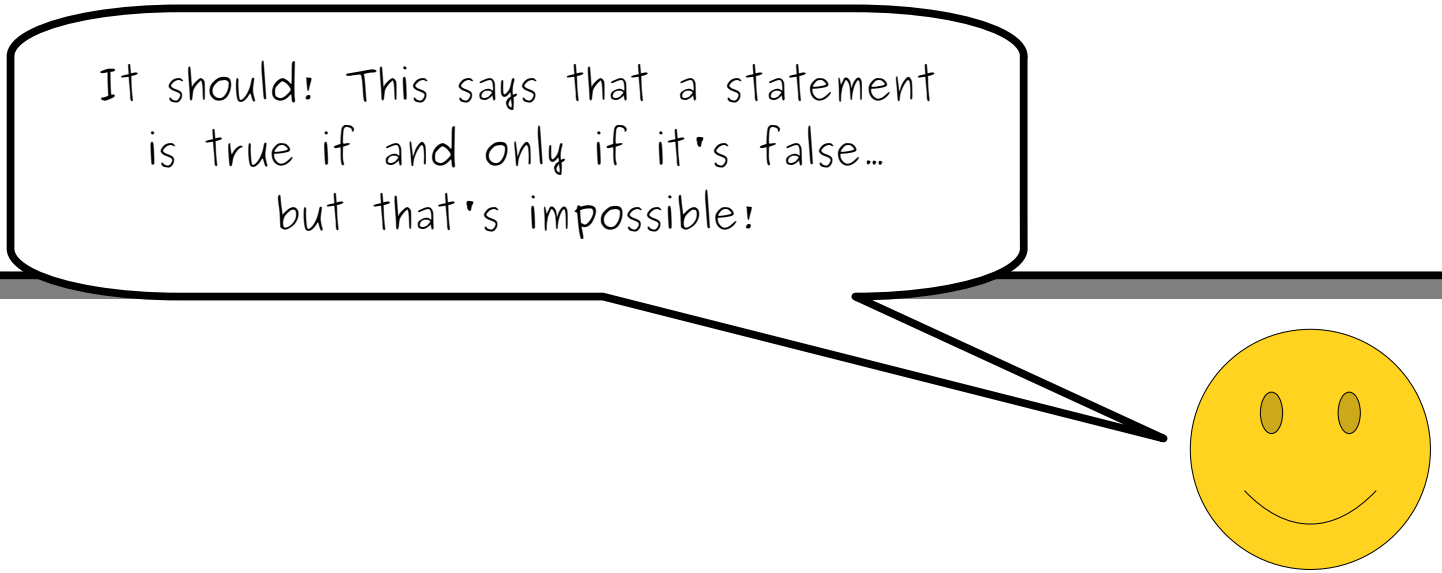

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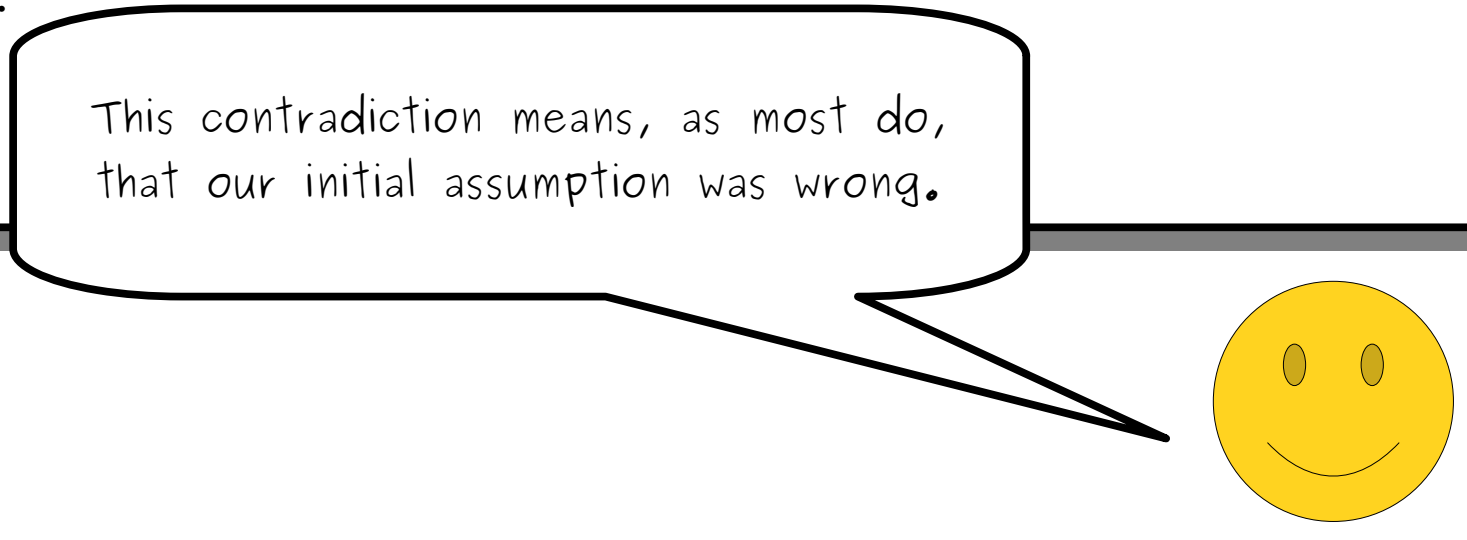

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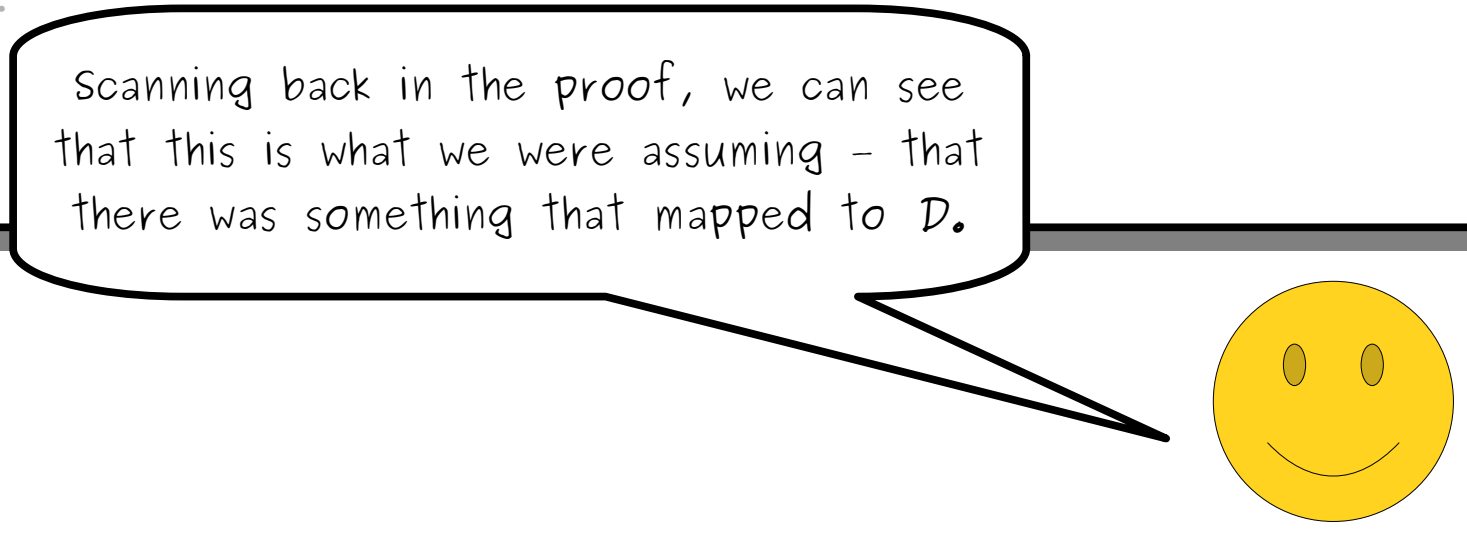

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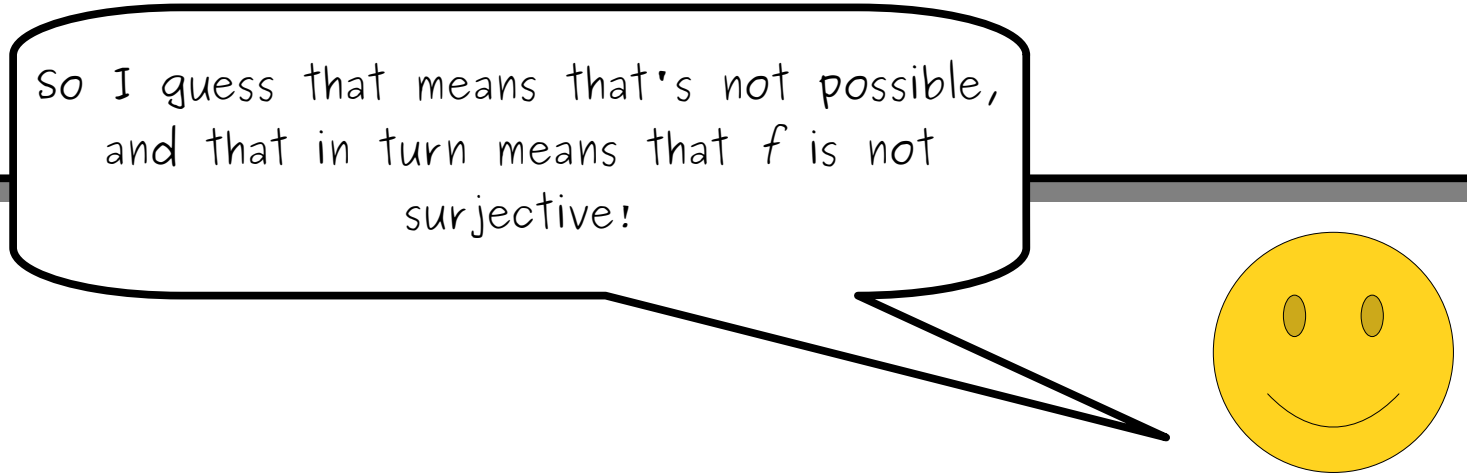

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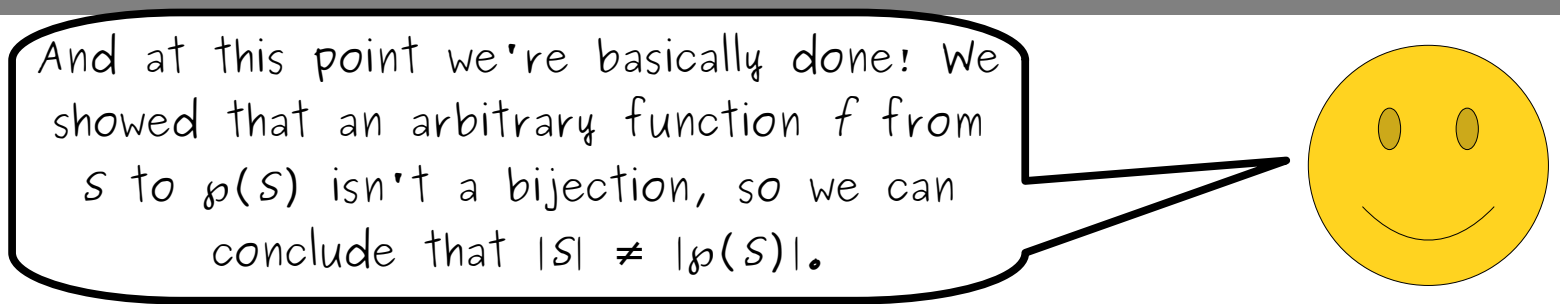

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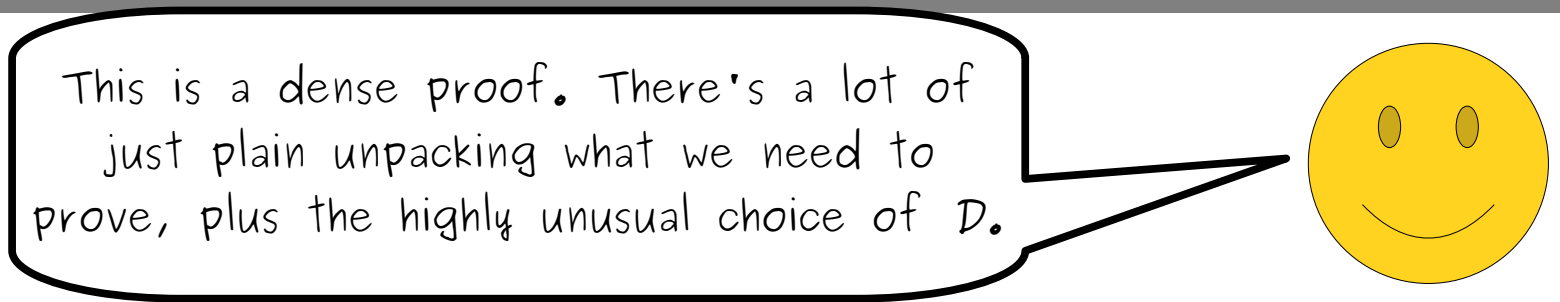

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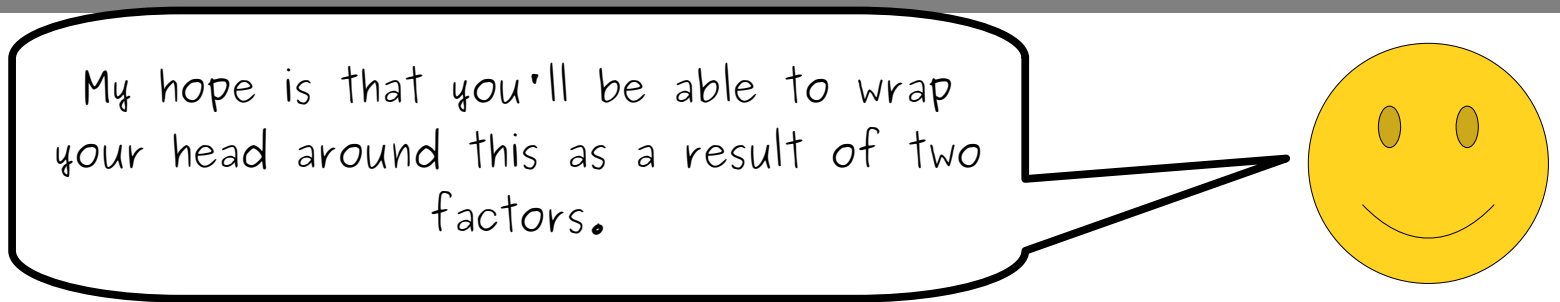

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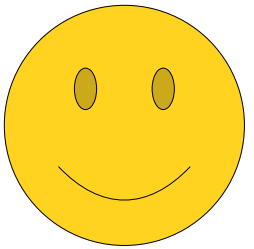
First, you have the visual intuition for the proof. I encourage you to look back at the visual and to make sure you see where these choices come from.



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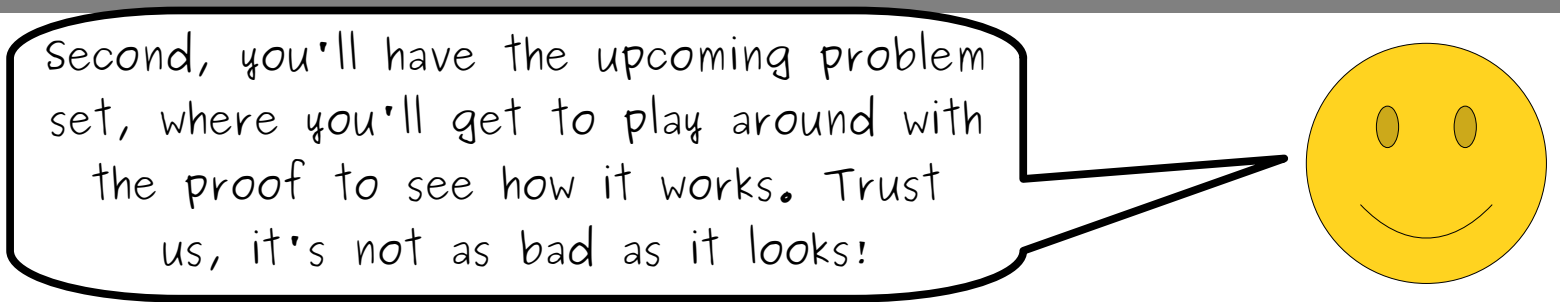
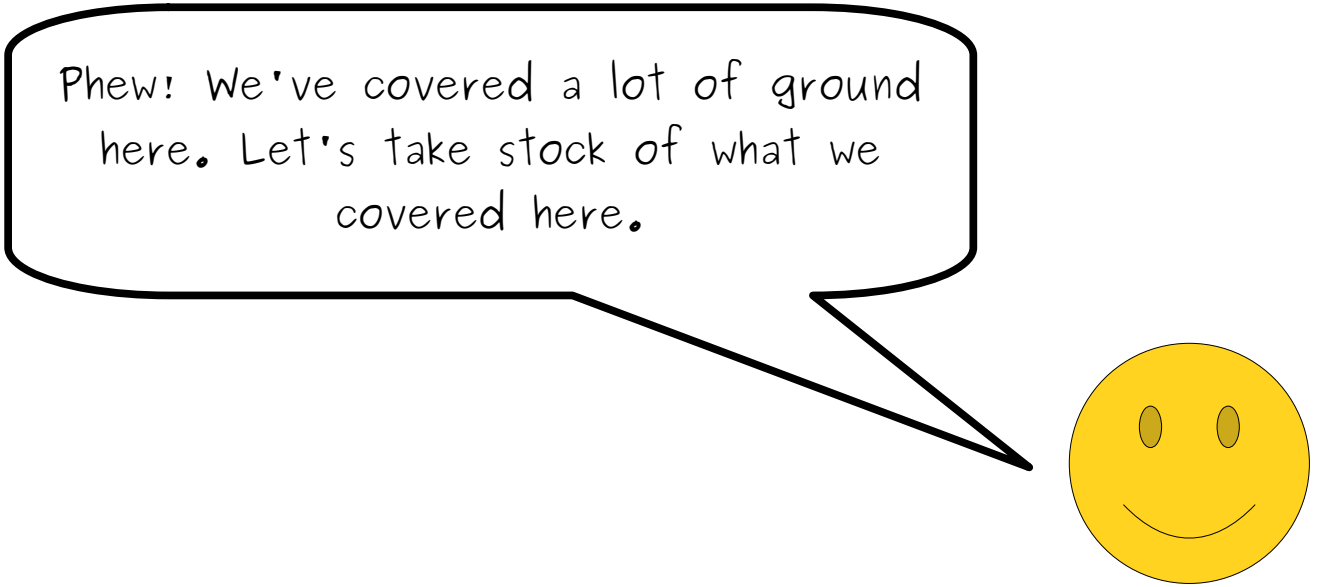
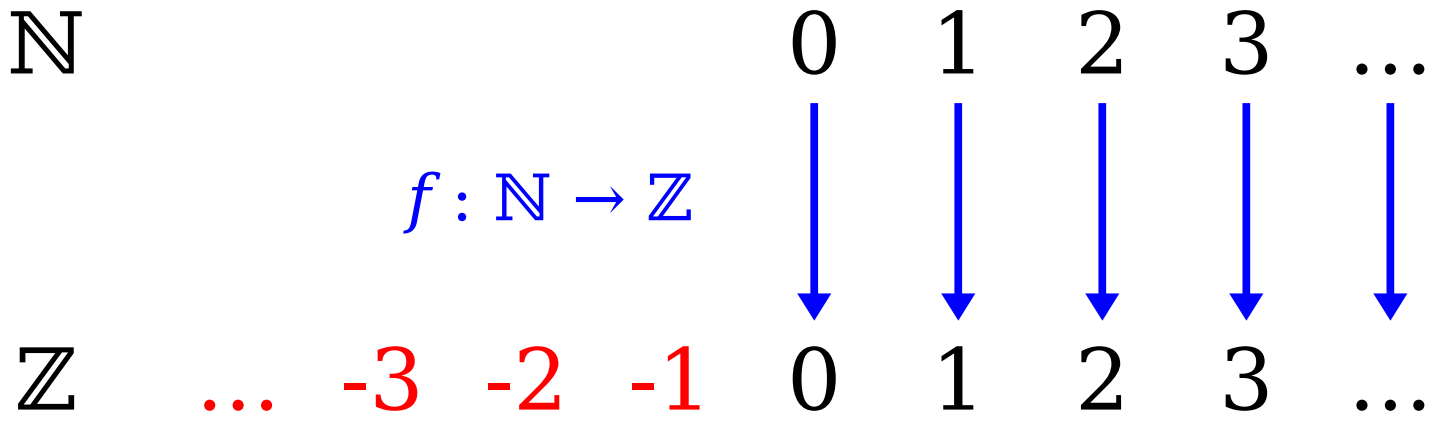



Phew! We've covered a lot of ground here. Let's take stock of what we covered here.



$|S| \leq |T|$ if there is an ***injective*** function $f : S \to T$

First, we came up with a rigorous definition of what $|S| \leq |T|$ means. This allows us to prove results about how set cardinalities rank against one another... something you (hypothetically speaking) might need for the upcoming problem set.

$|S| < |T|$ if $|S| \leq |T|$ and $|S| \neq |T|$

There is an injective function $f : S \to T$

No function $f : S \to T$ is a bijection.

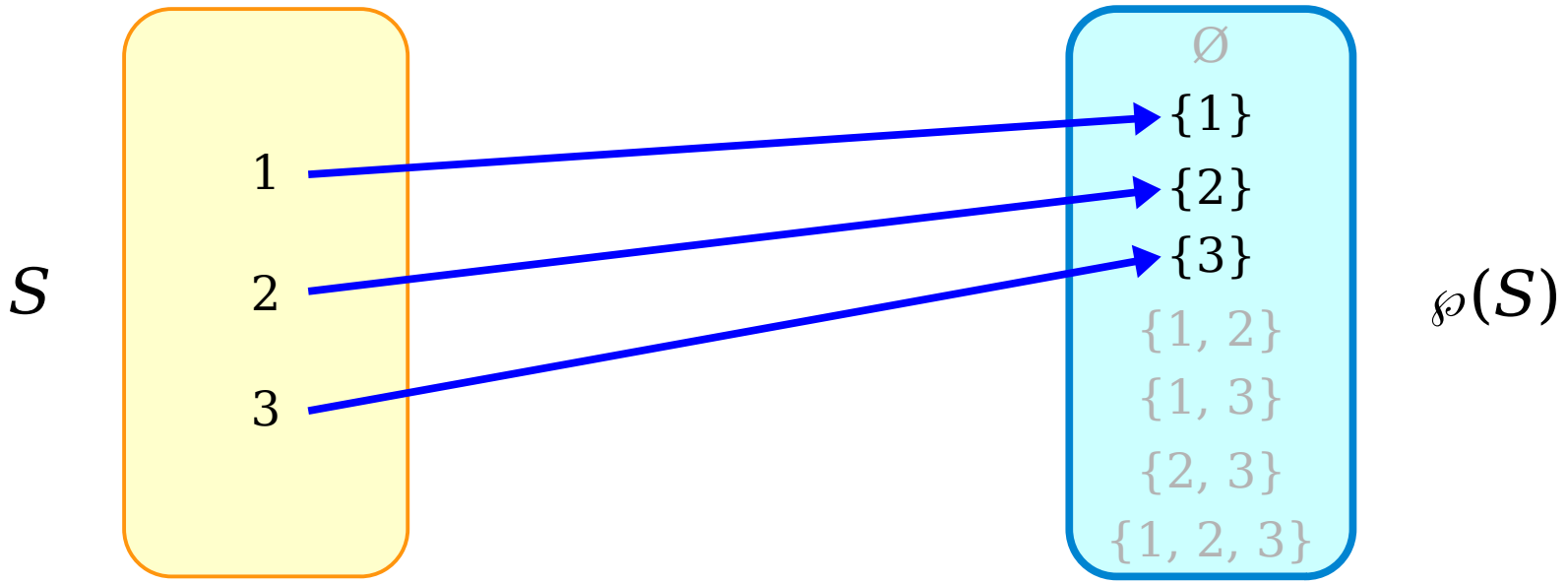
Second, we used that definition to come up with a definition for $|S| < |T|$, which lets us say that one set is strictly smaller than another.

There *is* an injection $f : S \to \wp(S)$
There is *no* bijection $f : S \to \wp(S)$

If S is a set, then $|S| < |\wp(S)|$

This in turn allowed us to determine what, specifically, we need to prove to prove Cantor's theorem.

Lemma 1: If S is a set, then there's an injection from S to $\wp(S)$.



$$f : S \to \wp(S)$$
$$f(x) = \{x\}$$

We saw how to find an injection from S to $\wp(S)$ for any set S.

Lemma 2: If $f : S \to \wp(S)$ is a function, then it is not bijective.

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longrightarrow \{ x_0, \quad x_2, \quad x_4, \quad \dots \}$

$x_1 \longrightarrow \{ x_0, \quad x_3, \quad x_4, \quad \dots \}$

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$\{ \quad x_1, \; x_2, \quad x_4, \quad \dots \}$

Set D

We used the visual intuition of the diagonal argument to precisely define the diagonal set.

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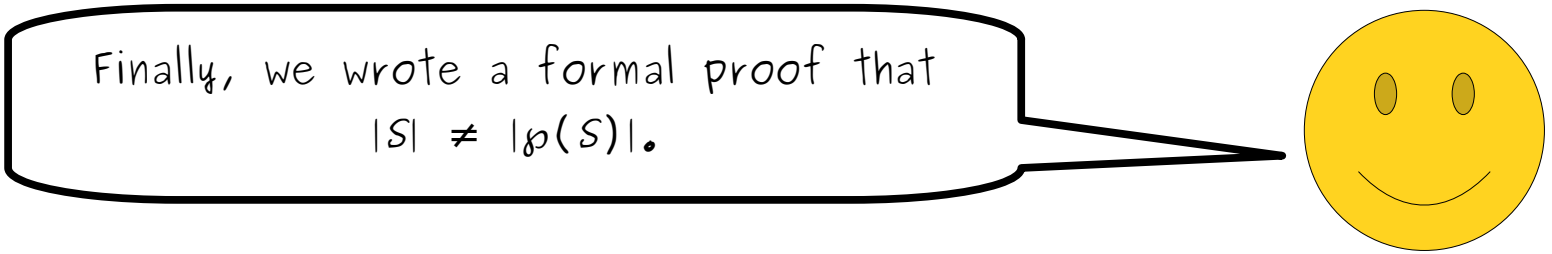

Hope this helps!
Please feel free to ask questions if you have them.

Did you find this useful? If so, let us know! We can go and make more guides like these.